ELEMENTS OF ALGEBRA.

BY

G. A. WENTWORTH, A.M.,

PROFESSOR OF MATHEMATICS IN PHILLIPS EXETER ACADEMY.

SHORTER COURSE.

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PREFACE.

THIS Shorter Course in Algebra is designed for schools that have not sufficient time for the author's full course. The book, however, contains a full treatment of the topics usually found in an elementary algebra. The rules are deduced from processes immediately preceding. Each principle is fully illustrated, and a sufficient number of well-graded problems is given to fix it firmly in the pupil's mind before he proceeds to another. Examples are worked out in order to exhibit the best methods of dealing with different classes of problems and the best arrangement of the work. Such aid is given in the statement of problems as experience has shown to be necessary for the best results.

It is presumed that pupils will have a fair acquaintance with Arithmetic before beginning the study of Algebra, and that sufficient time will be given to learn the language and the simple operations of algebra before the harder parts of the book are reached. "Make haste slowly" should be the watchword of the early chapters. Care has been taken to exclude all difficult problems, and yet the problems are not so easy as to deprive the student of a real satisfaction in mastering them.

Any corrections or suggestions relating to the work will be thankfully received.

G. A. WENTWORTH.

PHILLIPS EXETER ACADEMY,
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CHAPTER I.

Quantity and Number.

1. Whatever may be regarded as being made up of parts like the whole is called a **Quantity.**

2. To **measure** a quantity of any kind is to find how many times it contains another *known quantity of the same kind.*

3. A *known quantity* which is adopted as a standard for measuring quantities of the same kind is called a **Unit.** Thus, the foot, the pound, the dollar, the day, are units for measuring distance, weight, money, time.

4. A **Number** arises from the repetitions of the unit of measure, and shows *how many times* the unit is contained in the quantity measured.

5. When a quantity is measured, the result obtained is expressed by prefixing to the *name* of the unit the *number* which shows how many times the unit is contained in the quantity measured; and the two combined denote a quantity expressed in units. Thus, 7 feet, 8 pounds, 9 dollars, 10 days, are quantities expressed in their respective units.

When a question about a quantity includes the unit, the answer is a *number;* when it does not include the unit, the answer is a *quantity.* Thus, if a man who has fifteen bushels of wheat be asked *how many bushels* of wheat he has, the answer is the *number*, fifteen; if he be asked *how much* wheat he has, the answer is the *quantity*, fifteen bushels.

A number answers the question, How many? a quantity, the question, How much?

Numbers.

6. The symbols which **Arithmetic** employs to represent numbers are the figures 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. The *natural* series of numbers begins with 0; each succeeding number is obtained by adding one to the preceding number, and the series is infinite.

7. Besides figures, the chief symbols used in Arithmetic are:

$+$ (read, plus), the sign of addition.
$-$ (read, minus), the sign of subtraction.
$\times$ (read, multiplied by), the sign of multiplication.
$\div$ (read, divided by), the sign of division.
$=$ (read, is equal to), the sign of equality.

Exercise. — Read:

$7 + 12 = 19.$	$8 + 3 - 5 = 20 - 15 + 1.$
$9 - 4 = 5.$	$24 + 6 = 10 \times 3.$
$6 \times 4 = 24.$	$14 - 7 + 5 = 6 \times 2.$
$48 \div 3 = 16.$	$9 \times 5 = 180 \div 4.$

8. Any figure, or combination of figures, as 7, 28, 346, has one, and only one, value. That is, figures represent

particular numbers. But numbers possess many *general properties*, which are true, not only of a particular number, but of all numbers.

Thus, the sum of 12 and 8 is 20, and the difference between 12 and 8 is 4. Their sum added to their difference is 24, which is twice the greater number. Their difference taken from their sum is 16, which is twice the smaller number.

9. As this is true of any two numbers, we have this general property: *The sum of two numbers added to their difference is twice the greater number; the difference of two numbers taken from their sum is twice the smaller number.* Or,

1. (greater number + smaller number) + (greater number − smaller number) = twice greater number.
2. (greater number + smaller number) − (greater number − smaller number) = twice smaller number.

But these statements may be very much shortened; for, as greater number and smaller number may mean any two numbers, two letters, as a and b, may be used to represent them; and $2a$ may represent twice the greater, and $2b$ twice the smaller. Then these statements become:

1. $(a+b)+(a-b)=2a.$
2. $(a+b)-(a-b)=2b.$

In studying the general properties of numbers, letters may represent any numerical values consistent with the conditions of the problem.

10. It is also convenient to use letters to denote numbers which are *unknown*, and which are to be found from certain given relations to other known numbers.

Thus, the solution of the problem, "Find two numbers such that, when the greater is divided by the less, the quotient is 4, and the remainder 3; and when the sum of the two numbers is increased by 38, and the result divided by the greater of the two numbers, the quotient is 2 and the remainder 2," is much simplified by the use of letters to represent the unknown numbers.

11. The science which employs letters in reasoning about numbers, either to discover their *general properties*, or to find the value of an *unknown number* from its relations to known numbers, is called **Algebra.**

Algebraic Numbers.

12. There are quantities which stand to each other in such opposite relations that, when we combine them, they cancel each other entirely or in part. Thus, six dollars *gain* and six dollars *loss* just cancel each other; but ten dollars *gain* and six dollars *loss* cancel each other only in part. For the six dollars *loss* will cancel six dollars of the *gain* and will leave four dollars.

An opposition of this kind exists in *assets* and *debts*, in *income* and *outlay*, in motion *forwards* and *backwards*, in motion *to the right* and *to the left*, in time *before* and *after* a fixed date, in the degrees *above* and *below* zero on a thermometer.

From this relation of quantities a question often arises which is not considered in Arithmetic; namely, the subtracting of a greater number from a smaller. This cannot be done in Arithmetic, for the real nature of subtraction consists in *counting backwards*, along the natural series of numbers. If we wish to subtract four from six, we start at six in the natural series, count four units backwards, and

arrive at two, the difference sought. If we subtract six from six, we start at six in the natural series, count six units backwards, and arrive at zero. If we try to subtract nine from six, we cannot do it, because, when we have counted backwards as far as zero, *the natural series of numbers comes to an end.*

13. In order to subtract a greater number from a smaller it is necessary to *assume* a new series of numbers, beginning at zero and extending to the left of zero. The series to the left of zero must ascend from zero by the repetitions of the unit, precisely like the natural series to the right of zero; and the *opposition* between the right-hand series and the left-hand series must be clearly marked. This opposition is indicated by calling every number in the right-hand series a *positive* number, and prefixing to it, when written, the sign $+$; and by calling every number in the left-hand series a *negative* number, and prefixing to it the sign $-$. The two series of numbers will be written thus:

$$\ldots\ldots -4,\ -3,\ -2,\ -1,\quad 0,\ +1,\ +2,\ +3,\ +4, \ldots\ldots$$

If, now, we wish to subtract 9 from 6, we begin at 6 in the positive series, count nine units in the *negative direction* (to the left), and arrive at -3 in the negative series. That is, $6 - 9 = -3$.

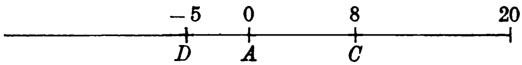
The result obtained by subtracting a greater number from a less, when both are positive, is *always a negative number.*

If a and b represent any two numbers of the positive series, the expression $a - b$ will denote a positive number when a is greater than b; will be equal to zero when a is equal to b; will denote a negative number when a is less than b.

If we wish to add 9 to -6, we begin at -6, in the

negative series, count nine units in the *positive direction* (to the right), and arrive at $+3$, in the positive series.

We may illustrate the use of positive and negative numbers as follows:



Suppose a person starting at A walks 20 feet to the right of A, and then returns 12 feet, where will he be? Answer: at C, a point 8 feet to the right of A. That is, 20 feet − 12 feet = 8 feet; or, $20 - 12 = 8$.

Again, suppose he walks from A to the right 20 feet, and then returns 20 feet, where will he be? Answer: at A, the point from which he started. That is, $20 - 20 = 0$.

Again, suppose he walks from A to the right 20 feet, and then returns 25 feet, where will he now be? Answer: at D, a point 5 feet to the left of A. That is, if we consider distance measured in feet to the left of A as forming a negative series of numbers, beginning at A, $20 - 25 = -5$. Hence, the phrase, 5 feet to the left of A, is now expressed by the negative number -5.

14. Numbers provided with the sign $+$ or $-$ are called **algebraic numbers.** They are unknown in Arithmetic, but play a very important part in Algebra. In contradistinction, numbers considered without regard to the signs $+$ or $-$ are termed **absolute numbers.**

15. Every algebraic number, as $+4$ or -4, consists of a sign $+$ or $-$ and the absolute value of the number; in this case 4. The sign shows whether the number belongs to the positive or negative series of numbers; the absolute value shows what place the number has in the positive or negative series.

16. When no sign stands before a number, the sign $+$ is always understood; thus, 4 means the same as $+4$, a means the same as $+a$. But *the sign* $-$ *is never omitted.*

17. Two numbers which have, one the sign $+$ and the other the sign $-$, are said to have **unlike signs.**

18. Two numbers which have the same absolute values, but unlike signs, always cancel each other when combined; thus, $+4-4=0$, $+a-a=0$.

19. The use of the signs $+$ and $-$, to indicate addition and subtraction, must be carefully distinguished from their use to indicate in which series, the positive or the negative, a given number belongs. In the first sense, they are signs of *operations*, and are common to both Arithmetic and Algebra. In the second sense, they are signs of *opposition*, and are employed in Algebra alone.

Factors and Powers.

20. When a number consists of the product of two or more numbers, each of these numbers is called a **factor** of the product.

When these numbers are denoted by letters, the sign $\times$ is omitted; thus, instead of $a \times b$, we write ab; instead of $a \times b \times c$, we write abc.

The expression abc must not be confounded with $a+b+c$; the first is a product, the second is a sum. If $a=2$, $b=3$, $c=4$, then

$$abc = 2 \times 3 \times 4 = 24;$$
$$a+b+c = 2+3+4 = 9.$$

21. Factors expressed by letters are called **literal** factors; factors expressed by figures are called **numerical** factors.

22. A known factor of a product which is prefixed to another factor, to show how many times that factor is taken, is called a **coefficient.** Thus, in $7c$, 7 is the coefficient of c; in $7ax$, 7 is the coefficient of ax, or, if a be known, $7a$ is the coefficient of x. When no numerical coefficient occurs in a product, 1 is always understood. Thus, ax means the same as $1ax$.

23. A product consisting of two or more **equal** factors is called a **power** of that factor.

24. The **index** or **exponent of a power** is a small figure or letter placed at the right of a number, to show how many times the number is taken as a factor. Thus, a^1, or simply a, denotes that a is taken once as a factor; a^2 denotes that a is taken twice as a factor; a^3 denotes that a is taken three times as a factor; and a^n denotes that a is taken n times as a factor. These are read: the first power of a; the second power of a; the third power of a; the nth power of a.

a^3 is written instead of aaa.

a^n is written instead of aaa, etc., repeated n times.

The meaning of coefficient and exponent must be carefully distinguished. Thus,

$$4a = a + a + a + a;$$
$$a^4 = a \times a \times a \times a.$$

If $a = 3$,
$$4a = 3 + 3 + 3 + 3 = 12;$$
$$a^4 = 3 \times 3 \times 3 \times 3 = 81.$$

25. The second power of a number is generally called the *square* of that number; thus, a^2 is called the *square* of a, because if a denote the number of units of length in the side of a square, a^2 denotes the number of units of surface in the square.

The third power of a number is generally called the *cube* of that number; thus, a^3 is called the *cube* of a, because if a denote the number of units of length in the edge of a cube, a^3 denotes the number of units of volume in the cube.

ALGEBRAIC SYMBOLS.

26. Known numbers in Algebra are denoted by figures and by the first letters of some alphabet; as, a, b, c, etc.; a', b', c', read *a prime*, *b prime*, *c prime*, etc.; a_1, b_1, c_1, read *a one*, *b one*, *c one*.

Unknown numbers are generally denoted by the last letters of some alphabet; as, x, y, z, x', y', z', etc.

27. The **symbols of operations** are the same in Algebra as in Arithmetic. One point of difference, however, must be carefully observed. When a symbol of operation is omitted in the notation of Arithmetic, it is always the *symbol of addition*; but when a symbol of operation is omitted in the notation of Algebra, it is always the *symbol of multiplication*. Thus, 456 means $400 + 50 + 6$, but $4ab$ means $4 \times a \times b$; $4\frac{5}{8}$ means $4 + \frac{5}{8}$, but $4\frac{a}{b}$ means $4 \times \frac{a}{b}$.

28. The **symbols of relation** are $=$, $>$, $<$, which stand for the words, "is equal to," "is greater than," and "is less than," respectively.

29. The **symbols of aggregation** are the bar, $|$; the vinculum, ——; the parenthesis, ( ); the bracket, [ ]; and the brace, { }. Thus, each of the expressions, $\left.\begin{array}{r} x \\ +y \end{array}\right|$, $\overline{x+y}$, $(x+y)$, $[x+y]$, $\{x+y\}$, signifies that $x+y$ is to be treated as a single number.

30. The **symbols of continuation** are dots, ..., or dashes, ------, and are read, "and so on."

31. The **symbol of deduction** is $\therefore$, and is read, "hence," or "therefore."

ALGEBRAIC EXPRESSIONS.

32. An **algebraic expression** is any number written in algebraic symbols. Thus, $8c$ is the algebraic expression for 8 times the number denoted by c.

$7a^2 - 3ab$ is the algebraic expression for 7 times the square of the number denoted by a, diminished by 3 times the product of the numbers denoted by a and b.

33. A **term** is an algebraic expression the parts of which are not separated by the sign of addition or subtraction. Thus, $3ab$, $5xy$, $3ab \div 5xy$, $3ab + 5xy$, are terms.

34. A **monomial** or **simple expression** is an expression which contains only one term. As, $3ab$.

35. A **polynomial** or **compound expression** is an expression which contains two or more terms. A **binomial** is a polynomial of two terms. As, $3ab + 5xy$. A **trinomial** is a polynomial of three terms. As, $3ab + 5xy + 2z$.

36. **Like terms** are terms which have the same letters, and the corresponding letters affected by the same exponents. Thus, $7a^2cx^3$ and $-5a^2cx^3$ are like terms; but $7a^2cx^3$ and $-5ac^2x^3$ are unlike terms.

37. The **dimensions** of a term are its literal factors.

38. The **degree** of a term is equal to the number of its dimensions, and is found by taking the sum of the exponents of its literal factors. Thus, $3xy$ is of the *second* degree, and $5x^2yz^3$ is of the *sixth* degree.

39. A polynomial is said to be **homogeneous** when all its terms are of the same degree. Thus, $7x^3 - 5x^2y + xyz$ is homogeneous, for each term is of the third degree.

40. A polynomial is said to be **arranged** according to the powers of some letter when the exponents of that letter either descend or ascend in order of magnitude. Thus, $3ax^3 - 4bx^2 - 6ax + 8b$ is arranged according to the descending powers of x, and $8b - 6ax - 4bx^2 + 3ax^3$ is arranged according to the ascending powers of x.

41. The **numerical value** of an algebraic expression is the number obtained by giving a particular value to each letter, and then performing the operations indicated.

42. Two numbers are **reciprocals** of each other when their product is equal to unity. Thus, a and $\frac{1}{a}$ are reciprocals.

Axioms.

43. 1. Things which are equal to the same thing are equal to each other.

2. If equal numbers be added to equal numbers, the sums will be equal.

3. If equal numbers be subtracted from equal numbers, the remainders will be equal.

4. If equal numbers be multiplied into equal numbers, the products will be equal.

5. If equal numbers be divided by equal numbers, the quotients will be equal.

6. If the same number be both added to and subtracted from another, the value of the latter will not be altered.

7. If a number be both multiplied and divided by another, the value of the former will not be altered.

Ex. 1.

If $a=1$, $b=2$, $c=3$, $d=4$, $e=5$, $f=0$, find the numerical values of the following expressions:

1. $9a+2b+3c-2f$.
2. $4e-3a-3b+5c$.
3. $8abc-bcd+9cde-def$.
4. $\frac{4ac}{b}+\frac{8bc}{d}-\frac{5cd}{e}$.
5. $7e+bcd-\frac{3bde}{2ac}$.
6. $abc^2+bcd^2-dea^2+f^3$.
7. $e^4+6e^2b^2+b^4-4e^3b-4eb^3$.
8. $\frac{8a^2+3b^2}{a^2b^2}+\frac{4c^2+6b^2}{c^2-b^2}-\frac{c^2+d^2}{e^2}$.
9. $\frac{b^2+d^2}{b^2+d^2-bd}$.
10. $\frac{e^3-d^3}{e^2+ed+d^2}$.

In simplifying compound expressions, each term must be reduced to its simplest form before the operations of addition and subtraction are performed.

Simplify the following expressions:

11. $100+80\div 4$.
12. $75-25\times 2$.
13. $25+5\times 4-10\div 5$.
14. $24-5\times 4\div 10+3$.
15. $(24-5)\times(4\div 10+3)$.

Find the numerical value of the following expressions, in which $a=2$, $b=10$, $x=3$, $y=5$:

16. $xy+4a\times 2$.
17. $xy-15b\div 5$.
18. $3x+7y\div 7$.
19. $6b-8y\div 2b$.
20. $(6b-8y)\div 2by+2b$.
21. $6b-(8y\div 2y)\times b-2b$.
22. $6b\div(b-y)-3x+bxy\div 10a$.

Ex. 2.

Algebraic Notation.

1. Express the sum of a and b.
2. Express the double of x.
3. By how much is a greater than 5?
4. If x be a whole number, express the next number above it.
5. Write five numbers in order of magnitude, so that x shall be the middle number.
6. What is the sum of $x + x + x + \cdots\cdots$ written a times?
7. If the product be xy and the multiplier x, what is the multiplicand?
8. A man who has a dollars spends b dollars; how many dollars has he left?
9. A regiment of men can be drawn up in a ranks of b men each, and there are c men over; of how many men does the regiment consist?
10. To-day the thermometer indicates m degrees above 0. Yesterday it indicated n degrees below 0. What is the variation in temperature between yesterday and to-day?
11. How many rolls of paper g feet long and h feet wide will be required to paper a wall a feet long and b feet high?
12. Write, the sum of x and y divided by c is equal to the product of a, b, and m, diminished by six times c, and increased by the quotient of a divided by the sum of x and y.
13. Write, six times the square of n, divided by m minus a, increased by five b into the expression c plus d minus a.

Ex. 3.

That the beginner may see how Algebra is employed in the solution of problems, the following simple exercises are introduced:

1. John and James together have \$6. James has twice as much as John. How much has each?

Let x denote the *number* of dollars John has.

Then $2x =$ *number* of dollars James has,

and $x + 2x =$ number of dollars both have.

But $6 =$ number of dollars both have.

Hence, $x + 2x = 6$,

or $3x = 6$,

and $x = 2$.

Therefore, John has \$2, and James has \$4.

2. A stick of timber 40 feet long is sawed in two, so that one part is two-thirds as long as the other. Required the length of each part.

Let $3x$ denote the *number* of feet in the longer part.

Then $2x =$ *number* of feet in the shorter part,

and $3x + 2x =$ number of feet in both together.

Hence, $3x + 2x = 40$,

or $5x = 40$,

and $x = 8$.

Therefore, the longer part, or $3x$, is 24 feet long; and the shorter, or $2x$, is 16 feet.

NOTE. The *unit* of the quantity sought is always given, and only the **number** of such units is required. Therefore, x must never be put for *money*, *length*, *time*, *weight*, etc., but always for the required **number** *of specified units* of money, length, time, weight, etc.

The beginner should give particular attention to this caution.

3. If a number is multiplied by 8, the product is 248. What is the number?
4. The greater of two numbers is six times the smaller, and their sum is 35. Required the numbers.
5. Thomas had 75 cents. After spending a part of his money, he found he had twice as much left as he had spent. How much had he spent?
6. A tree 75 feet high was broken, so that the part broken off was four times the length of the part left standing. Required the length of each part.
7. Four times the smaller of two numbers is three times the greater, and their sum is 63. Find the numbers.
8. A farmer sold a sheep, a cow, and a horse, for $216. He sold the cow for seven times as much as the sheep, and the horse for four times as much as the cow. How much did he get for each?
9. George bought some apples, pears, and oranges, for 91 cents. He paid twice as much for the pears as for the apples, and twice as much for the oranges as for the pears. How much money did he spend for each?
10. A man bought a horse, wagon, and harness, for $350. He paid for the horse four times as much as for the harness, and for the wagon one-half as much as for the horse. What did he pay for each?
11. Distribute $3 among Thomas, Richard, and Henry, so that Thomas and Richard shall each have twice as much as Henry.
12. Three men, A, B, and C, pay $1000 taxes. B pays 4 times as much as A, and C pays as much as A and B together. How much does each pay?
13. The age of a boy is three times the age of his sister, and their ages together are 16 years. What is the age of each?

CHAPTER II.

ADDITION AND SUBTRACTION.

44. AN algebraic number which is to be added or subtracted is often inclosed in a parenthesis, in order that the signs $+$ and $-$ which are used to distinguish positive and negative numbers may not be confounded with the $+$ and $-$ signs that denote the operations of addition and subtraction. Thus, $+4+(-3)$ expresses the sum, and $+4-(-3)$ expresses the difference, of the numbers $+4$ and -3.

45. In order to add two algebraic numbers, we begin at *the place in the series which the first number occupies*, and count, *in the direction indicated by the sign of the second number*, as many units as are equal to the absolute value of the second number. Thus, the sum of $+4+(+3)$ is found by counting from $+4$ three units in the *positive* direction, and is, therefore, $+7$; the sum of $+4+(-3)$ is found by counting from $+4$ three units in the *negative* direction, and is, therefore, $+1$.

In like manner, the sum of $-4+(+3)$ is -1, and the sum of $-4+(-3)$ is -7. That is,

(1) $+4+(+3)=7$; (3) $-4+(+3)=-1$;

(2) $+4+(-3)=1$; (4) $-4+(-3)=-7$.

I. Therefore, to add two numbers with **like** signs, *find the* **sum** *of their absolute values, and prefix the common sign to the sum.*

II. To add two numbers with **unlike** signs, *find the* **difference** *of their absolute values, and prefix the sign of the greater number to the difference.*

Ex. 4.

1. $+16+(-11)=$
2. $-15+(-25)=$
3. $+68+(-79)=$
4. $-\ 7+(+\ 4)=$
5. $+33+(+18)=$
6. $+378+(+709)+(-592)=$
7. A man has $5242 and owes $2758. How much is he worth?
8. The First Punic War began B.C. 264, and lasted 23 years. When did it end?
9. Augustus Cæsar was born B.C. 63, and lived 77 years. When did he die?
10. A man goes 65 steps forwards, then 37 steps backwards, then again 48 steps forwards. How many steps did he take in all? How many steps is he from where he started?

Addition of Monomials.

46. If a and b denote the absolute values of any two numbers, 1, 2, 3, 4 (§ 45) become:

(1) $+a+(+b)=a+b$; (3) $-a+(+b)=-a+b$;

(2) $+a+(-b)=a-b$; (4) $-a+(-b)=-a-b$.

Therefore, to add two terms, *write them one after the other with unchanged signs.*

It should be noticed that the order of the terms is immaterial. Thus, $+a-b=-b+a$. If $a=8$ and $b=12$, the result in either case is -4.

47.

$$3a+5a+2a+6a+a=17a.$$
$$-2c-3c-c-4c-8c=-18c.$$

Therefore, to add several **like terms** which have the **same**

sign, *add the coefficients, prefix the common sign, and annex the common symbols.*

48. $7a - 6a + 11a + a - 5a - 2a = 19a - 13a = 6a.$

$-3a - 15a - 7a + 14a - 2a = 14a - 27a = -13a.$

Therefore, to add several **like terms** which have **not all the same sign,** *find the difference between the sum of the positive coefficients and the sum of the negative coefficients, prefix the sign of the greater sum, and annex the common symbols.*

49. $5a - 2b + 3a = 8a - 2b.$

$-3ax + 8y + 9ax - 4c = 6ax + 8y - 4c.$

Therefore, to add terms which are **not all like terms,** *combine the like terms, and write down the other terms, each preceded by its proper sign.*

Ex. 5.

1. $5ab + (-5ab) =$
2. $8mx + (-2mx) =$
3. $-13mng + (-7mng) =$
4. $-5x^2 + (+8x^2) =$
5. $25my^2 + (-18my^2) =$
6. $7ab + (-5ab) =$
7. $120my + (-95my) =$
8. $-33ab^2 + (11ab^2) =$
9. $-75xy + (+20xy) =$
10. $+15a^2x^2 + (-a^2x^2) =$
11. $-b^2m^3 + (+7b^2m^3) =$
12. $5a + (-3b) + (+4a) + (-7b) =$
13. $4a^2c + (-10xyz) + (+6a^2c) + (-9xyz) + (-11a^2c) + (+20xyz) =$
14. $3x^2y + (-4ab) + (-2mn) + (+5x^2y) + (-x^2y) + (-4x^2y) =$

Addition of Polynomials.

50. Two or more polynomials are added by adding their separate terms.

It is convenient to arrange the terms in columns, so that like terms shall stand in the same column. Thus,

$$
\begin{array}{rrrr}
(1)\ 2a^3 & -3a^2b & +4ab^2 & +\ b^3 \\
a^3 & +4a^2b & -7ab^2 & -2b^3 \\
-3a^3 & +\ a^2b & -3ab^2 & -4b^3 \\
2a^3 & +2a^2b & +6ab^2 & -3b^3 \\
\hline
2a^3 & +4a^2b & & -8b^3
\end{array}
\qquad
\begin{array}{rrrr}
(2)\ -2x^2y & & +6y^3 & -1 \\
-4x^2y & +2xy^2 & & +5 \\
6x^2y & & & +2 \\
x^2y & & -\ y^3 & \\
-2x^2y & & & -5 \\
\hline
-\ x^2y & +2xy^2 & +5y^3 & +1
\end{array}
$$

Ex. 6.

Add:

1. $5a+3b+c,\ 3a+3b+3c,\ a+3b+5c.$

2. $7a-4b+c,\ 6a+3b-5c,\ -12a+4c.$

3. $a+b-c,\ b+c-a,\ c+a-b,\ a+b-c.$

4. $a+2b+3c,\ 2a-b-2c,\ b-a-c,\ c-a-b.$

5. $a-2b+3c-4d,\ 3b-4c+5d-2a,$ $5c-6d+3a-4b,\ 7d-4a+5b-4c.$

6. $x^3-4x^2+5x-3,\ 2x^3-7x^2-7x^2-14x+5,$ $-x^3+9x^2+x+8.$

7. $x^4-2x^3+3x^2,\ x^3+x^2+x,\ 4x^4+5x^3,$ $2x^2+3x-4,\ -3x^2-2x-5.$

8. $a^3+3ab^2-3a^2b-b^3,\ 2a^3+5a^2b-6ab^2-7b^2,$ $a^3-ab^2+2b^3.$

9. $2ab-3ax^2+2a^2x,\ 12ab-6a^2x+10ax^2,$ $ax^3-8ab-5a^2x.$

10. $c^4-3c^3+2c^2-4c+7,\ 2c^4+3c^3+2c^2+5c+6,$
$-4c^4-4c^2-5.$

11. $3x^2-xy+xz-3y^2+4yz-z^2,\ -5x^2-xy-xz+5yz,$
$6x^2-6y-6z,\ 4yz-5yz+3z^2,$
$-4x^2+y^2+3yz+3z^2.$

12. $m^5-3m^4n-6m^3n^2,\ +m^3n^2+m^2n^3-5m^4n,$
$7m^3n^2+4m^2n^3-3mn^4,\ -2m^2n^3-3mn^4+4n^5,$
$2mn^4+2n^5+3m^5,\ -n^5+2m^5+7m^4n.$

SUBTRACTION.

51. In order to find the difference between two algebraic numbers, we begin *at the place in the series which the minuend occupies*, and *count in the direction opposite to that indicated by the sign of the subtrahend* as many units as are equal to the absolute value of the subtrahend.

Thus, the difference between $+4$ and $+3$ is found by counting from $+4$ three units in the *negative* direction, and is, therefore, $+1$; the difference between $+4$ and -3 is found by counting from $+4$ three units in the *positive* direction, and is, therefore, $+7$.

In like manner, the difference between -4 and $+3$ is -7; the difference between -4 and -3 is -1.

Compare these results with results obtained in addition:

(1) $+4-(+3)=1$	$+4+(-3)=1.$
(2) $+4-(-3)=7$	$+4+(+3)=7.$
(3) $-4-(+3)=-7$	$-4+(-3)=-7.$
(4) $-4-(-3)=-1$	$-4+(+3)=-1.$

Or, (1) $+4-(+3)=+4+(-3).$

(2) $+4-(-3)=+4+(+3).$

(3) $-4-(+3)=-4+(-3).$

(4) $-4-(-3)=-4+(+3).$

52. From (1) and (3), it is evident that *subtracting a positive number is equivalent to adding an equal negative number.*

From (2) and (4), it is evident that *subtracting a negative number is equivalent to adding an equal positive number.*

To subtract, therefore, one algebraic number from another, *change the sign of the subtrahend, and then add the subtrahend to the minuend.*

Ex. 7.

1. $+25-(+16)=$
3. $-\ 31-(+58)=$
2. $-50-(-25)=$
4. $+107-(-93)=$
5. Rome was ruled by emperors from B.C. 30 to its fall, A.D. 476. How long did the empire last?
6. The continent of Europe lies between 36° and 71° north latitude, and between 12° west and 63° east longitude (from Paris). How many degrees does it extend in latitude, and how many in longitude?

Subtraction of Monomials.

If a and b denote the absolute values of any two numbers, 1, 2, 3, and 4 (§ 51) become:

(1) $+a-(+b)=a-b.$ (3) $-a-(+b)=-a-b.$

(2) $+a-(-b)=a+b.$ (4) $-a-(-b)=-a+b.$

To subtract, therefore, one term from another, *change the sign of the term to be subtracted and write the terms one after the other.*

Ex. 8.

1. $5x - (-4x) =$
2. $-3ab - (+5ab) =$
3. $3ab^2 - (+10ab^2) =$
4. $15m^2x^2 - (-7m^2x^2) =$
5. $-7ay - (-3ay) =$
6. $17ax^3 - (-24ax^3) =$
7. $5a^2x - (-3a^2x) =$
8. $-4xy - (-5xy) =$
9. $8ax - (-3ay) =$
10. $2ab^2y - (+aby) =$
11. $9x^2 + (5x^2) - (+8x^2) =$
12. $5x^2y - (-18x^2y) + (-10x^2y) =$
13. $17ax^3 - (-ax^3) - (+24ax^3) =$
14. $-3ab + (2mx) - (-4mx) =$
15. $3a - (+2b) - (-4c) =$

Subtraction of Polynomials.

53. When one polynomial is to be subtracted from another, place its terms under the like terms of the other, change the signs of the subtrahend, and add.

$$\begin{array}{ll} \text{From} & 4x^3 - 3x^2y - xy^2 + 2y^3 \\ \text{take} & \underline{2x^3 - x^2y + 5xy^2 - 3y^3} \end{array}$$

Change the signs of the subtrahend and add:

$$\begin{array}{r} 4x^3 - 3x^2y - xy^2 + 2y^3 \\ \underline{-2x^3 + x^2y - 5xy^2 + 3y^3} \\ 2x^3 - 2x^2y - 6xy^2 + 5y^3 \end{array}$$

$$\begin{array}{lr} \text{From} & a^3x^2 + 2a^2x^3 - 4ax^4 \\ \text{take} & \underline{a^5 + 4a^3x^2 - 3a^2x^3 - 4ax^4} \\ & -a^5 - 3a^3x^2 + 5a^2x^3 \end{array}$$

In the last example we have conceived the signs to be changed without actually changing them. The beginner should do the examples by both methods until he has acquired sufficient practice, when he should use the second method only.

Ex. 9.

1. From $6a-2b-c$ take $2a-2b-3c$.

2. From $3a-2b+3c$ take $2a-7b-c-b$.

3. From $7x^2-8x-1$ take $5x^2-6x+3$.

4. From $4x^4-3x^3-2x^2-7x+9$ take $x^4-2x^3-2x^2+7x-9$.

5. From $2x^2-2ax+3a^2$ take x^2-ax+a^2.

6. From $x^2-3xy-y^2+yz-2z^2$ take $x^2+2xy+5xz-3y^2-2z^2$.

7. From $a^3-3a^2b+3ab^2-b^3$ take $-a^3+3a^2b-3ab^2+b^3$.

8. From $x^2-5xy+xz-y^2+7yz+2z^2$ take $x^2-xy-xz+2yz+3z^2$.

9. From $2ax^2+3abx-4b^2x+12b^3$ take $ax^2-4abx+bx^2-5b^2x-x^3$.

10. From $6x^3-7x^2y+4xy^2-2y^3-5x^2+xy-4y^2+2$ take $8x^3-7x^2y+xy^2-y^3+9x^2-xy+6y^2-4$.

11. From a^4-b^4 take $4a^3b-6a^2b^2+4ab^3$, and from the result take $2a^4-4a^3b+6a^2b^2+4ab^3-2b^4$.

12. From $x^3y^2-3x^2y^3+4xy^4-y^5$ take $-x^5+2x^4y-4xy^4-4y^5$. Add the same two expressions, and subtract the former result from the latter.

13. From $a^2b^2-a^2bc-8ab^2c-a^2c^2+abc^2-6b^2c^2$ take $2a^2bc-5ab^2c+2abc^2-5b^2c^2$.

14. From $12a+3b-5c-2d$ take $10a-b+4c-3d$, and show that the result is numerically correct when $a=6$, $b=4$, $c=1$, $d=5$.

15. What number must be added to a to make b; and what number must be taken from $2a^3-6a^2b+6ab^2-2b^3$ to leave $a^3-7a^2b-3b^3$?

16. From $2x^2-y^2-2xy+z^2$ take $x^2-y^2+2xy-z^2$.

17. From $12ac+8cd-9$ take $-7ac-9cd+8$.

18. From $-6a^2+2ab-3c^2$ take $4a^2+6ab-4c^2$.

19. From $9xy-4x-3y+7$ take $8xy-2x+3y+6$.

20. From $-a^2bc-ab^2c+abc^2-abc$ take $a^2bc+ab^2c-abc^2+abc$.

21. From $7x^2-2x+4$ take $2x^2+3x-1$.

22. From $3x^2+2xy-y^2$ take $-x^2-3xy+3y^2$, and from the remainder take $3x^2+4xy-5y^2$.

23. From ax^2-by^2 take cx^2-dy^2.

24. From $ax+bx+by+cy$ take $ax-bx-by+cy$.

25. From $5x^2+4x-4y+3y^2$ take $5x^2-3x+3y+y^2$.

26. From $a^2b^2+12abc-9ax^2$ take $4ab^2-6acx+3a^2x$.

27. From $a^2-2ab+c^2-3b^2$ take $2a^2-2ab+3b^2$.

28. From the sum of the first four of the following expressions, $a^2+b^2+c^2+d^2$, $d^2+b^2+c^2$, $a^2-c^2+b^2-d^2$, $a^2-b^2+c^2+d^2$, $b^2+c^2+d^2-a^2$, take the sum of the last four.

29. From $2x^2-2y^2-z^2$ take $3y^2+2x^2-z^2$, and from the remainder take $3z^2-2y^2-x^2$.

30. From $a^3-2a^2c+3ac^2$ take the sum of $a^2c-2a^3+2ac^2$ and $a^3-ac^2-a^2c$.

PARENTHESES.

54. From (§ 52), it appears that

$$
\begin{aligned}
&(1)\quad a+(+b)=a+b.\\
&(2)\quad a+(-b)=a-b.\\
&(3)\quad a-(+b)=a-b.\\
&(4)\quad a-(-b)=a+b.
\end{aligned}
$$

The same laws respecting the removal of parentheses hold true whether one or more terms are inclosed. Hence, when an expression within a parenthesis is preceded by a **plus sign,** the parenthesis may be removed.

When an expression within a parenthesis is preceded by a **minus sign,** the parenthesis may be removed *if the sign of every term within the parenthesis be changed.* Thus:

$$
\begin{aligned}
&(1)\quad a+(b-c)=a+b-c.\\
&(2)\quad a-(b-c)=a-b+c.
\end{aligned}
$$

55. Expressions may occur with more than one parenthesis. These parentheses may be removed in succession, by removing *first, the innermost parenthesis;* next, the innermost of all that remain, and so on. Thus:

$$
\begin{aligned}
(1)\quad & a-\{b-(c-d)\}\\
&=a-\{b-c+d\}\\
&=a-b+c-d.
\end{aligned}
$$

$$
\begin{aligned}
(2)\quad & a-[b-\{c+(d-\overline{e-f})\}]\\
&=a-[b-\{c+(d-e+f)\}]\\
&=a-[b-\{c+d-e+f\}]\\
&=a-[b-c-d+e-f]\\
&=a-b+c+d-e+f.
\end{aligned}
$$

Ex. 10.

Simplify the following expressions by removing the parentheses and combining like terms.

1. $(a+b)+(b+c)-(a+c)$.
2. $(2a-b-c)-(a-2b+c)$.
3. $(2x-y)-(2y-z)-(2z-x)$.
4. $(a-x-y)-(b-x+y)+(c+2y)$.
5. $(2x-y+3z)+(-x-y-4z)-(3x-2y-z)$.
6. $(3a-b+7c)-(2a+3b)-(5b-4c)+(3c-a)$.
7. $1-(1-a)+(1-a+a^2)-(1-a+a^2-a^3)$.
8. $a-\{2b-(3c+2b)-a\}$.
9. $2a-\{b-(a-2b)\}$.
10. $3a-\{b+(2a-b)-(a-b)\}$.
11. $7a-[3a-\{4a-(5a-2a)\}]$.
12. $2x+(y-3z)-\{(3x-2y)+z\}+5x-(4y-3z)$.
13. $\{(3a-2b)+(4c-a)\}-\{a-(2b-3a)-c\}$ $+\{a-(b-5c-a)\}$.
14. $a-[2a+(3a-4a)]-5a-\{6a-[(7a+8a)-9a]\}$.
15. $2a-(3b+2c)-[5b-(6c-6b)+5c$ $-\{2a-(c+2b)\}]$.
16. $a-[2b+\{3c-3a-(a+b)\}+\{2a-(b+c)\}]$.
17. $16-x-[7x-\{8x-(9x-\overline{3x-6x})\}]$.
18. $2a-[3b+(2b-c)-4c+\{2a-(3b-\overline{c-2b})\}]$.
19. $a-[2b+\{3c-3a-(a+b)\}+2a-(b+3c)]$.
20. $a-[5b-\{a-(3c-3b)+2c-(a-2b-c)\}]$.

56. The rules for introducing parentheses follow directly from the rules for removing them:

1. Any number of terms of an expression may be put within a parenthesis, and the sign **plus** placed before the whole.

2. Any number of terms of an expression may be put within a parenthesis, and the sign **minus** placed before the whole; *provided the sign of every term within the parenthesis be changed.*

It is usual to prefix to the parenthesis the sign of the first term that is to be inclosed within it.

Ex. 11.

Express in binomials, and also in trinomials:

1. $2a - 3b - 4c + d + 3e - 2f.$

2. $a - 2x + 4y - 3z - 2b + c.$

3. $a^5 + 3a^4 - 2a^3 - 4a^2 + a - 1.$

4. $-3a - 2b + 2c - 5d - e - 2f.$

5. $ax - by - cz - bx + cy + az.$

6. $2x^5 - 3x^4y + 4x^3y^2 - 5x^2y^3 + xy^4 - 2y^5.$

7. Express each of the above in trinomials, having the last two terms inclosed by *inner* parentheses.

Collect in parentheses the coefficients of x, y, z in

8. $2ax - 6ay + 4bz - 4bx - 2cx - 3cy.$

9. $ax - bx + 2ay + 3y + 4az - 3bz - 2z.$

10. $ax - 2by + 5cz - 4bx - 3cy + az - 2cx - ay + 4bz.$

11. $12ax + 12ay + 4by - 12bz - 15cx + 6cy + 3cz.$

12. $2ax - 3by - 7cz - 2bx + 2cx + 8cz - 2cx - cy - cz.$

CHAPTER III.

Multiplication of Algebraic Numbers.

57. The operation of finding the sum of 3 numbers, each equal to 5, is symbolized by the expression, $3 \times 5 = 15$, read, "three times five is equal to fifteen"; or, by the expression $5 \times 3 = 15$, read, "five multiplied by three is equal to fifteen."

58. With reference to this operation, this sum is called the **product**; one of the equal numbers is called the **multiplicand**; and the number which shows how many times the multiplicand is to be taken is called the **multiplier.**

59. The multiplier means so many **times.** The multiplicand can be a *positive* or a *negative number;* but the multiplier can only mean that the multiplicand is taken *so many times to be added*, or *so many times to be subtracted.*

60. If we have to multiply 867 by 98, we may put the multiplier in the form $100 - 2$. The 100 will mean that the multiplicand is taken 100 times *to be added;* the -2 will mean that the multiplicand is taken twice *to be subtracted.*

In general, a multiplier with $+$ before it, expressed or understood, means that the multiplicand is taken so many times *to be added;* and a multiplier with $-$ before it means that the multiplicand is taken so many times *to be subtracted.* Thus,

(1) $+3 \times (+5) = (+5) + (+5) + (+5)$, or $(+15)$.

(2) $+3 \times (-5) = (-5) + (-5) + (-5)$, or (-15).

(3) $-3 \times (+5) = -(+5) - (+5) - (+5)$, or (-15).

(4) $-3 \times (-5) = -(-5) - (-5) - (-5)$, or $(+15)$.

From these four cases it follows, that, in finding the product of two numbers,

61. *Like* signs produce **plus**; *unlike* signs, **minus**.

Ex. 12.

1. $-17 \times 8 =$
2. $-12.8 \times 25 =$
3. $-3.29 \times 5.49 =$
4. $-18 \times -5 =$
5. $43 \times -6 =$
6. $457 \times 100 =$
7. $(-358 - 417) \times -79 =$
8. $(7.512 - \{-2.894\}) \times (-6.037 + \{13.963\}) =$

62. The product of more than two factors, each preceded by $-$, will be positive or negative, according as the number of such factors is even or odd. Thus,

$$-2 \times -3 \times -4 = +6 \times -4 = -24.$$

$$-2 \times -3 \times -4 \times -5 = -24 \times -5 = +120.$$

9. $13 \times 8 \times -7 =$
10. $-38 \times 9 \times -6 =$
11. $-20.9 \times -1.1 \times 8 =$
12. $-78.3 \times -0.57 \times +1.38 \times -27.9 =$
13. $-2.906 \times -2.076 \times -1.49 \times 0.89 =$

Multiplication of Monomials.

63. The product of numerical factors is a new number in which no trace of the original factors is found. Thus, $4 \times 9 = 36$. But the product of literal factors can only be expressed by writing them one after the other. Thus, the product of a and b is expressed by ab; the product of ab and cd is expressed by $abcd$.

64. If we have to multiply $5a$ by $-4b$, the factors will give the same result in whatever order they are taken. Thus, $5a \times -4b = 5 \times -4 \times a \times b = -20 \times ab = -20ab$.

65. Hence, to find the product of monomials, *annex the literal factors to the product of the numerical factors.*

66. $a^2 \times a^3 = aa \times aaa = aaaaa = a^5$.

$a^2 \times a^3 \times a^4 = aa \times aaa \times aaaa = aaaaaaaaa = a^9$.

It is evident that the exponent of the product is equal to the sum of the exponents of the factors. Hence,

67. *The product of two or more powers of any number is that number with an exponent equal to the sum of the exponents of the factors.*

Ex. 13.

1. $+a \times +b = +ab$.
2. $+a \times -b = -ab$.
3. $-a \times +b = -ab$.
4. $-a \times -b = +ab$.
5. $7a \times 5b = 35ab$.
6. $-3p \times 8m = -24pm$.
7. $3a^2 \times -a^3 = -3a^5$.
8. $-3a \times 2a^5 = -6a^6$.
9. $6a \times -2a =$
10. $5mn \times 9m =$

11. $3ax \times -4by =$

12. $-8cm \times dn =$

13. $-7ab \times 2ac =$

14. $5m^2x \times 3mx^2 =$

15. $5a^m \times -2a^n =$

16. $3a^2x^2 \times 7a^3x^4 =$

17. $7a \times -4b \times -8c =$

18. $8ab^2 \times 3ac \times -4c^2 =$

19. $27ab \times -39mp \times 18ap =$

20. $6ab^2y^3 \times 2b^3y^3 \times -5a^2y =$

21. $7m^2x \times 3mx^2 \times -2mq =$

22. $-3pq^2 \times 6p^3q \times 8p^2q^3 =$

23. $2a^2m^3x^4 \times 3am^5x^2 \times 4a^3mx^2 =$

24. $6x^2yz^3 \times -9x^2y^2z^2 \times -3x^4yz =$

25. $3ax \times 2am \times -4mx \times b^2 =$

26. $7am^2 \times 3b^2n^2 \times -4ab \times a^2bn \times -2b^2n \times -mn^2 =$

OF POLYNOMIALS BY MONOMIALS.

68. If we have to multiply $a + b$ by n, that is, to take $(a + b)$ n times to be added, we have,

$$\begin{aligned}(a+b) \times n &= (a+b)+(a+b)+(a+b) \ldots\ldots n \text{ times} \\ &= a+a+a \ldots\ldots n \text{ times} + b+b+b \ldots\ldots n \text{ times} \\ &= a \times n + b \times n \\ &= an + bn.\end{aligned}$$

As it is immaterial in what order the factors are taken,

$$n \times (a+b) = an + bn.$$

In like manner,

$$(a+b+c) \times n = an + bn + cn,$$

or,

$$n(a+b+c) = an + bn + cn.$$

Hence, to multiply a polynomial by a monomial,

69. *Multiply each term of the polynomial by the monomial, and add the partial products.*

Ex. 14.

1. $(6a-5b)\times 3c = 18ac - 15bc.$
2. $(2+3a-4a^2-5a^3)6a^2 = 12a^2+18a^3-24a^4-30a^5.$
3. $5a(3b+4c-d) = 15ab+20ac-5ad.$
4. $-3ax(-by-2cz+5) = 3abxy+6acxz-15ax.$
5. $(4a^2-3b)\times 3ab =$
6. $(8a^2-9ab)\times 3a^2 =$
7. $(3x^2-4y^2+5z^2)\times 2x^2y =$
8. $(a^3x-5a^2x^2+ax^3+2x^4)\times ax^2y =$
9. $(-9a^5+3a^3b^2-4a^2b^3-b^5)\times -3ab^4 =$
10. $(3x^3-2x^2y-7xy^2+y^3)\times -5x^2y =$
11. $(-4xy^2+5x^2y+8x^3)\times -3x^2y =$
12. $(-3+2ab+a^2b^2)\times -a^4 =$
13. $(-z-2xz^2+5x^2yz^2-6x^3y^2+3x^3y^2z)\times -3x^3yz =$

Of Polynomials by Polynomials.

70. If we have $a+b+c$ to be multiplied by $m+n+p$, we may represent the multiplicand $a+b+c$ by M. Then,

$$M(m+n+p) = M\times m + M\times n + M\times p.$$

If now we substitute for M its value,

$$\begin{aligned}(a+b+c)(m+n+p) &= (a+b+c)\times m\\ &+ (a+b+c)\times n\\ &+ (a+b+c)\times p;\end{aligned}$$

or, $$\begin{aligned}(a+b+c)(m+n+p) = {} & am + bm + cm \\ & + an + bn + cn \\ & + ap + bp + cp.\end{aligned}$$

That is, to find the product of two polynomials,

71. *Multiply the multiplicand by each term of the multiplier and add the partial products; or, multiply each term of one factor by each term of the other, and add the partial products.*

72. In multiplying polynomials, it is a convenient arrangement to write the multiplier under the multiplicand, and place like terms of the partial products in columns.

Thus:

(1)
$$\begin{array}{rrr} 5a & -\ 6b & \\ 3a & -\ 4b & \\ \hline 15a^2 & -\ 18ab & \\ & -\ 20ab & +\ 24b^2 \\ \hline 15a^2 & -\ 38ab & +\ 24b^2 \end{array}$$

(2) Multiply $4x+3+5x^2-6x^3$ by $4-6x^2-5x$.

Arrange both multiplicand and multiplier according to the ascending powers of x.

$$\begin{array}{rrrrrr} 3 & +\ 4x & +\ 5x^2 & -\ 6x^3 & & \\ 4 & -\ 5x & -\ 6x^2 & & & \\ \hline 12 & +\ 16x & +\ 20x^2 & -\ 24x^3 & & \\ & -\ 15x & -\ 20x^2 & -\ 25x^3 & +\ 30x^4 & \\ & & -\ 18x^2 & -\ 24x^3 & -\ 30x^4 & +\ 36x^5 \\ \hline 12 & +\ x & -\ 18x^2 & -\ 73x^3 & & +\ 36x^5 \end{array}$$

(3) Multiply $1+2x+x^4-3x^2$ by x^3-2-2x.

Arrange according to the descending powers of x.

$$\begin{array}{llllllll} x^4 & -\ 3x^2 & +\ 2x & +\ 1 & & & & \\ x^3 & -\ 2x & -\ 2 & & & & & \\ \hline x^7 & -\ 3x^5 & +\ 2x^4 & +\ x^3 & & & & \\ & -\ 2x^5 & & +\ 6x^3 & -\ 4x^2 & -\ 2x & & \\ & & -\ 2x^4 & & +\ 6x^2 & -\ 4x & -\ 2 & \\ \hline x^7 & -\ 5x^5 & & +\ 7x^3 & +\ 2x^2 & -\ 6x & -\ 2 & \end{array}$$

(4) Multiply $a^2+b^2+c^2-ab-bc-ac$ by $a+b+c$.
Arrange according to descending powers of a.

$$
\begin{array}{llllllll}
a^2 & -ab & -ac & +b^2 & -bc & +c^2 & & \\
a & +b & +c & & & & & \\
\hline
a^3 & -a^2b & -a^2c & +ab^2 & -abc & +ac^2 & & \\
 & +a^2b & & -ab^2 & -abc & & +b^3-b^2c & +bc^2 \\
 & & +a^2c & & -abc & -ac^2 & +b^2c & -bc^2+c^3 \\
\hline
a^3 & & & & -3abc & & +b^3 & +c^3
\end{array}
$$

The student should observe that, with a view to bringing like terms of the partial products in columns, the terms of the multiplicand and multiplier are arranged in the **same order.**

In order to test the accuracy of the work, interchange the multiplicand and multiplier. The result should be the same in both operations.

Ex. 15.

Multiply:

1. x^2-4 by x^2+5.
3. $a^4+a^2x^2+x^4$ by a^2-x^2.
2. $y-6$ by $y+13$.
4. x^2+xy+y^2 by $x-y$.
5. $2x-y$ by $x+2y$.
6. $2x^3+4x^2+8x+16$ by $3x-6$.
7. x^3+x^2+x-1 by $x-1$.
8. x^2-3ax by $x+3a$.
9. $2b^2+3ab-a^2$ by $-5b+7a$.
10. $2a+b$ by $a+2b$.
12. a^2-ab+b^2 by $a+b$.
11. a^2+ab+b^2 by $a-b$.
13. $2ab-5b^2$ by $3a^2-4ab$.
14. $-a^3+2a^2b-b^3$ by $4a^2+8ab$.
15. a^2+ab+b^2 by a^2-ab+b^2.

16. $a^3-3a^2b+3ab^2-b^3$ by $a^2-2ab+b^2$.

17. $x+2y-3z$ by $x-2y+3z$.

18. $2x^2+3xy+4y^2$ by $3x^2-4xy+yz$.

19. x^2+xy+y^2 by x^2+xz+z^2.

20. $a^2+b^2+c^2-ab-ac-bc$ by $a+b+c$.

21. $x^2-xy+y^2+x+y+1$ by $x+y-1$.

Arrange the multiplicand and multiplier according to the descending powers of a common letter, and multiply:

22. $5x+4x^2+x^3-24$ by $x^2+11-4x$.

23. $x^3+11x-4x^2-24$ by x^2+5+4x.

24. $x^4+x^2-4x-11+2x^3$ by x^2-2x+3.

25. $-5x^4-x^2-x+x^5+13x^3$ by x^2-2-2x.

26. $3x+x^3-2x^2-4$ by $2x+4x^3+3x^2+1$.

27. $5a^4+2a^2b^2+ab^3-3a^3b$ by $5a^3b-2ab^3+3a^2b^2+b^4$.

28. $4a^7y-32ay^7-8a^5y^3+16a^3y^5$ by $a^6y^2+4a^2y^6+4a^4y^4$.

29. $3m^3+3n^3+9mn^2+9m^2n$ by $6m^2n^3-2mn^4-6m^3n^2+2m^4n$.

30. $6a^5b+3a^2b^4-2ab^5+b^6$ by $4a^4-2ab^3-3b^4$.

Find the products of:

31. $x-3$, $x-1$, $x+1$, and $x+3$.

32. x^2-x+1, x^2+x+1, and x^4-x^2+1.

33. a^2+ab+b^2, a^2-ab+b^2, and $a^4-a^2b^2+b^4$.

34. $4a^3-4a^2b+ab^2$, $4a^2+3ab+b^2$, and $2a^2b+b^3$.

35. $x+a$, $x+2a$, $x-3a$, $x-4a$, and $x+5a$.

36. $9a^2+b^2$, $27a^3-b^3$, $27a^3+b^3$, and $81a^4-9a^2b^2+b^4$.

37. From the product of $y^2-2yz-z^2$ and $y^2+2yz-z^2$ take the product of $y^2-yz-2z^2$ and $y^2+yz-2z^2$.

38. Find the dividend when the divisor $=3a^2-ab-3b^2$, the quotient $=a^2b-2b^3$, the remainder $=-2ab^4-6b^5$.

The multiplication of polynomials may be *indicated* by inclosing each in a parenthesis and writing them one after the other. When the operations indicated are actually performed, the expression is said to be *simplified.*

Simplify:

39. $15x^2+24y^2-(3x+2y)(5x+6y)$.

40. $(a+b)(b+c)-(c+d)(d+a)-(a+c)(b-d)$.

41. $12ab+5b^2-(3a-4b)(2a+8b)-(3a-2)(a-b)$.

42. $(a+b+c)^2-a(b+c-a)-b(a+c-b)-c(a+b-c)$.

43. $(a-b)x-(b-c)a-\{(b-x)(b-a)-(b-c)(b+c)\}$.

44. $(m+n)m-\{(m-n)^2-(n-m)n\}$.

45. $\{ac-(a-b)(b+c)\}-b\{b-(a-c)\}$.

46. $(x-1)(x-2)-3x(x+3)+2\{(x+2)(x+1)-3\}$.

47. $4(a-3b)(a+3b)-2(a-6b)^2-2(a^2+6b^2)$.

48. $(x+y+z)(x+y-z)-(x-y+z)(-x+y+z)$.

49. $(15a^2-12b)(2a+3b)-(4a^2+15b)(5a^2-4b)$.

50. $(9a^2-4b^2)(a-1)-(3a-2b)(3a^2+ab-3a-2b)$.

51. $15x^2+24y^2-(3x+2y)(5x+6y)$.

52. $26ab-(9a-8b)(5a+2b)-(4b-3a)(15a+4b)$.

73. There are some examples in multiplication which occur so often in algebraical operations that they should be carefully noticed and remembered. The three which follow are of great importance:

$$
\begin{array}{lll}
(1)\quad \begin{array}{r} a + b \\ a + b \\ \hline a^2 + ab \\ ab + b^2 \\ \hline a^2 + 2ab + b^2 \end{array}
&
(2)\quad \begin{array}{r} a - b \\ a - b \\ \hline a^2 - ab \\ - ab + b^2 \\ \hline a^2 - 2ab + b^2 \end{array}
&
(3)\quad \begin{array}{r} a + b \\ a - b \\ \hline a^2 + ab \\ - ab - b^2 \\ \hline a^2 \qquad - b^2 \end{array}
\end{array}
$$

From (1) we have $(a+b)^2 = a^2 + 2ab + b^2$. That is,

74. *The square of the sum of two numbers is equal to the sum of their squares + twice their product.*

From (2) we have $(a-b)^2 = a^2 - 2ab + b^2$. That is,

75. *The square of the difference of two numbers is equal to the sum of their squares − twice their product.*

From (3) we have $(a+b)(a-b) = a^2 - b^2$. That is,

76. *The product of the sum and difference of two numbers is equal to the difference of their squares.*

77. A general truth expressed by symbols is called a **formula.**

78. By using the double sign $\pm$, read plus or minus, we may represent (1) and (2) by a single formula; thus,

$$(a \pm b)^2 = a^2 \pm 2ab + b^2;$$

in which expression the upper signs correspond with one another, and the lower with one another.

By remembering these formulas the square of any binomial, or the product of the sum and difference of any two numbers, may be written by inspection; thus:

Ex. 16.

1. $(127)^2 - (123)^2 = (127 + 123)(127 - 123)$
$= 250 \times 4 = 1000.$

2. $(29)^2 = (30 - 1)^2 = 900 - 60 + 1 = 841.$

3. $(53)^2 = (50 + 3)^2 = 2500 + 300 + 9 = 2809.$

4. $(3x + 2y)^2 = 9x^2 + 12xy + 4y^2.$

5. $(2a^2x - 5x^2y)^2 = 4a^4x^2 - 20a^2x^3y + 25x^4y^2.$

6. $(3ab^2c + 2a^2c^2)(3ab^2c - 2a^2c^2) = 9a^2b^4c^2 - 4a^4c^4.$

7. $(x + y)^2 =$
8. $(y - z)^2 =$
9. $(2x + 1)^2 =$
10. $(2a + 5b)^2 =$
11. $(1 - x^2)^2 =$
12. $(3ax - 4x^2)^2 =$
13. $(1 - 7a)^2 =$
14. $(5xy + 2)^2 =$
15. $(ab + cd)^2 =$
16. $(3mn - 4)^2 =$
17. $(12 + 5x)^2 =$
18. $(4xy^2 - yz^2)^2 =$
19. $(3abc - bcd)^2 =$
20. $(4x^3 - xy^2)^2 =$
21. $(x + y)(x - y) =$
22. $(2a + b)(2a - b) =$
23. $(3 - x)(3 + x) =$
24. $(3ab + 2b^2)(3ab - 2b^2) =$
25. $(4x^2 - 3y^2)(4x^2 + 3y^2) =$
26. $(a^3x^2 - by^4)(a^3x^2 + by^4) =$
27. $(6xy - 5y^2)(6xy + 5y^2) =$
28. $(4x^5 - 1)(4x^5 + 1) =$
29. $(1 + 3ab^3)(1 - 3ab^3) =$
30. $(ax + by)(ax - by)(a^2x^2 + b^2y^2) =$

79. Also the square of a trinomial should be carefully noticed.

$$\begin{array}{llllll} a+ & b+ & c & & & \\ a+ & b+ & c & & & \\ \hline a^2+ & ab+ & ac & & & \\ & ab & & +b^2+ & bc & \\ & & ac & + & bc+ & c^2 \\ \hline \end{array}$$

$$a^2+2ab+2ac+b^2+2bc+c^2$$
$$=a^2+b^2+c^2+2ab+2ac+2bc.$$

It is evident that this result is composed of two sets of numbers:

I. The squares of a, b, and c;

II. Twice the products of a, b, and c taken two and two.

Again,

$$\begin{array}{llllll} a- & b- & c & & & \\ a- & b- & c & & & \\ \hline a^2- & ab- & ac & & & \\ - & ab & & +b^2+ & bc & \\ & - & ac & + & bc+ & c^2 \\ \hline \end{array}$$

$$a^2-2ab-2ac+b^2+2bc+c^2$$
$$=a^2+b^2+c^2-2ab-2ac+2bc.$$

The law of formation is the same as before:

I. The squares of a, b, and c;

II. Twice the products of a, b, and c taken two and two.

The sign of each double product is + or − according as the signs of the factors composing it are *like* or *unlike*.

The same law holds good for the square of expressions containing more than three terms, and may be stated thus:

80. *To the sum of the squares of the several terms add twice the product of each term by each of the terms that follow it.*

By remembering this formula, the square of any polynomial may be written by inspection; thus:

Ex. 17.

1. $(x+y+z)^2=$
2. $(x-y+z)^2=$
3. $(m+n-p-q)^2=$
4. $(x^2+2x-3)^2=$
5. $(x^2+y^2-z^2)^2=$
6. $(x^4-4x^2y^2+y^4)^2=$
7. $(a^3+b^3+c^3)^2=$
8. $(x^3-y^3-z^3)^2=$
9. $(x+2y-3z)^2=$
10. $(x^2+2x-2)^2=$

81. Likewise, the product of two binomials of the form $x+a$, $x+b$ should be carefully noticed and remembered.

(1)
$$\begin{array}{r} x+5 \\ x+3 \\ \hline x^2+5x \qquad \\ 3x+15 \\ \hline x^2+8x+15 \end{array}$$

(3)
$$\begin{array}{r} x+5 \\ x-3 \\ \hline x^2+5x \qquad \\ -3x-15 \\ \hline x^2+2x-15 \end{array}$$

(2)
$$\begin{array}{r} x-5 \\ x-3 \\ \hline x^2-5x \qquad \\ -3x+15 \\ \hline x^2-8x+15 \end{array}$$

(4)
$$\begin{array}{r} x-5 \\ x+3 \\ \hline x^2-5x \qquad \\ +3x-15 \\ \hline x^2-2x-15 \end{array}$$

It will be observed that:

I. In all the results the first term is x^2 and the last term is the product of 5 and 3.

II. From (1) and (2), when the second terms of the binomials have *like* signs, the product has

the last term *positive;*

the *coefficient* of the middle term = the **sum** of 3 and 5;

the *sign* of the middle term is the same as that of the 3 and 5.

III. From (3) and (4), when the second terms of the binomials have *unlike* signs, the product has

the last term *negative;*

the *coefficient* of the middle term = the **difference** of 3 and 5;

the *sign* of the middle term is that of the **greater** of the two numbers.

82. These results may be deduced from the general formula,

$$(x+a)(x+b)=x^2+(a+b)x+ab,$$

by supposing for (1) a and b both positive;
(2) a and b both negative;
(3) a positive, b negative, and $a > b$;
(4) a negative, b positive, and $a > b$.

By remembering this formula the product of two binomials may be written by inspection; thus:

Ex. 18.

1. $(x+2)(x+3)=$
2. $(x+1)(x+5)=$
3. $(x-3)(x-6)=$
4. $(x-8)(x-1)=$
5. $(x-8)(x+1)=$
6. $(x-2)(x+5)=$
7. $(x-3)(x+7)=$
8. $(x-2)(x-4)=$
9. $(x+1)(x+11)=$
10. $(x-2a)(x+3a)=$
11. $(x-c)(x-d)=$
12. $(x-4y)(x+y)=$
13. $(a-2b)(a-5b)=$
14. $(x^2+2y^2)(x^2+y^2)=$

83. The second, third, and fourth powers of $a+b$ are found in the following manner:

$$\begin{array}{rllllll}
 & a & + & b & & & \\
 & a & + & b & & & \\
\hline
 & a^2 & + & ab & & & \\
 & & & ab & + & b^2 & \\
\hline
(a+b)^2 = & a^2 & + & 2ab & + & b^2 & \\
 & a & + & b & & & \\
\hline
 & a^3 & + & 2a^2b & + & ab^2 & \\
 & & & a^2b & + & 2ab^2 & + b^3 \\
\hline
(a+b)^3 = & a^3 & + & 3a^2b & + & 3ab^2 & + b^3 \\
 & a & + & b & & & \\
\hline
 & a^4 & + & 3a^3b & + & 3a^2b^2 & + ab^3 \\
 & & & a^3b & + & 3a^2b^2 & + 3ab^3 + b^4 \\
\hline
(a+b)^4 = & a^4 & + & 4a^3b & + & 6a^2b^2 & + 4ab^3 + b^4
\end{array}$$

From these results it will be observed that:

I. The number of terms is greater by one than the exponent of the power to which the binomial is raised.

II. In the first term, the exponent of a is the same as the exponent of the power to which the binomial is raised; and it decreases by one in each succeeding term.

III. b appears in the second term with 1 for an exponent, and its exponent increases by one in each succeeding term.

IV. The coefficient of the first term is 1.

V. The coefficient of the second term is the same as the exponent of the power to which the binomial is raised.

VI. The coefficient of each succeeding term is found from the next preceding term by multiplying its coefficient by the exponent of a, and dividing the product by a number greater by one than the exponent of b.

84. If b be negative, the terms in which the odd powers of b occur are negative. Thus:

$$(a-b)^4 = a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4.$$

Ex. 19.

Write the results:

1. $(x+a)^3=$	**17.** $(m+n)^{11}=$	**33.** $(x-1)^{11}=$
2. $(x-a)^3=$	**18.** $(p+v)^{13}=$	**34.** $(y-1)^8=$
3. $(x+1)^3=$	**19.** $(a-b)^4=$	**35.** $(z+1)^7=$
4. $(x-1)^3=$	**20.** $(a-b)^5=$	**36.** $(z-2)^8=$
5. $(x+a)^4=$	**21.** $(a-x)^7=$	**37.** $(2+x)^3=$
6. $(x-a)^4=$	**22.** $(h+z)^{13}=$	**38.** $(2-x)^4=$
7. $(x+1)^4=$	**23.** $(h-z)^{11}=$	**39.** $(3+x)^8=$
8. $(x-1)^4=$	**24.** $(a-c)^8=$	**40.** $(x-3)^9=$
9. $(x+y)^5=$	**25.** $(b-p)^9=$	**41.** $(z+5)^6=$
10. $(x-y)^5=$	**26.** $(z-d)^{13}=$	**42.** $(a-5)^8=$
11. $(x+1)^5=$	**27.** $(1+x)^{12}=$	**43.** $(4-b)^7=$
12. $(x-1)^5=$	**28.** $(1-x)^9=$	**44.** $(h-2)^{10}=$
13. $(b+x)^4=$	**29.** $(1-a)^8=$	**45.** $(a-10)^6=$
14. $(d+y)^5=$	**30.** $(1+a)^{15}=$	**46.** $(b+4)^5=$
15. $(c+z)^6=$	**31.** $(1-z)^{14}=$	**47.** $(2+x)^5=$
16. $(a+p)^7=$	**32.** $(x+1)^9=$	**48.** $(3-z)^5=$

CHAPTER IV.

Division.

85. Division is the operation by which, when a product and one of its factors are given, the other factor is determined.

86. With reference to this operation the product is called the **dividend**; the given factor the **divisor**; and the required factor the **quotient**.

87. The operation of division is indicated by the sign $\div$; by the colon :, or by writing the dividend over the divisor with a line drawn between them. Thus, $12 \div 4$, $12 : 4$, $\frac{12}{4}$, each means that 12 is to be divided by 4.

88. $+12$ divided by $+4$ gives the quotient $+3$; since only a positive number, $+3$, when multiplied by $+4$, can give the positive product, $+12$. § 61.

$+12$ divided by -4 gives the quotient -3; since only a negative number, -3, when multiplied by -4, can give the positive product, $+12$. § 61.

-12 divided by $+4$ gives the quotient -3; since only a negative number, -3, when multiplied by $+4$, can give the negative product, -12. § 61.

-12 divided by -4 gives the quotient $+3$; since only a positive number, $+3$, when multiplied by -4, can give the negative product, -12. § 61.

(1) $\frac{+12}{+4} = +3.$ (3) $\frac{-12}{+4} = -3.$

(2) $\frac{+12}{-4} = -3.$ (4) $\frac{-12}{-4} = +3.$

From (1) and (4) it follows that

89. The quotient is **positive** when the dividend and divisor have *like* signs.

From (2) and (3) it follows that

The quotient is **negative** when the dividend and divisor have *unlike* signs.

90. The **absolute value** of the quotient is equal to the quotient of the **absolute values** of the dividend and divisor.

Ex. 20.

1. $\frac{+264}{+4} =$
2. $\frac{-4648}{-8} =$
3. $\frac{+3840}{-30} =$
4. $\frac{-2568}{+12} =$
5. $\frac{106.33}{-4.9} =$
6. $\frac{-42.435}{+34.5} =$
7. $\frac{-264}{+24} =$
8. $\frac{-3670}{-85} =$
9. $\frac{+6.8503}{-61} =$
10. $\frac{-7.1560}{+324} =$
11. $\frac{-1}{-3.14159} =$
12. $\frac{-.31831}{-31.4159} =$

DIVISION OF MONOMIALS.

91. If we have to divide abc by bc, $aabx$ by aby, $12abc$ by $-4ab$, we write them as follows:

$$\frac{abc}{bc}=a; \qquad \frac{aabx}{aby}=\frac{ax}{y}; \qquad \frac{12abc}{-4ab}=-3c.$$

Hence, to divide one monomial by another,

92. *Write the dividend over the divisor with a line between them; if the expressions have common factors, remove the common factors.*

If we have to divide a^5 by a^2, a^6 by a^4, a^4 by a, we write them as follows:

$$\frac{a^5}{a^2}=\frac{aaaaa}{aa}=aaa=a^3;$$

$$\frac{a^6}{a^4}=\frac{aaaaaa}{aaaa}=aa=a^2;$$

$$\frac{a^4}{a}=\frac{aaaa}{a}=aaa=a^3.$$

93. That is, if a power of a number be divided by a *lower* power of the same number, *the quotient is that power of the number whose exponent is equal to the exponent of the dividend — that of the divisor.*

Again,

$$\frac{a^2}{a^5}=\frac{aa}{aaaaa}=\frac{1}{aaa}=\frac{1}{a^3};$$

$$\frac{a^3}{a^5}=\frac{aaa}{aaaaa}=\frac{1}{aa}=\frac{1}{a^2};$$

$$\frac{a^4}{a^8}=\frac{aaaa}{aaaaaaaa}=\frac{1}{aaaa}=\frac{1}{a^4}.$$

94. That is, if any power of a number be divided by a *higher* power of the same number, *the quotient is expressed by* 1 *divided by the number with an exponent equal to the exponent of the divisor* − *that of the dividend.*

Ex. 21.

1. $\frac{+ab}{+a} = +b.$
2. $\frac{+ab}{-a} = -b.$
3. $\frac{-ab}{+a} = -b.$
4. $\frac{-ab}{-a} = +b.$
5. $\frac{6mx}{2x} =$
6. $\frac{12a^4}{-3a} =$
7. $\frac{10ab}{2bc} =$
8. $\frac{x^3}{-x^5} =$
9. $\frac{-12am}{-2m} =$
10. $\frac{35abcd}{5bd} =$
11. $\frac{abx}{5aby} =$
12. $\frac{27a^7}{-3a^3} =$
13. $\frac{-3bmx}{4ax^2} =$
14. $\frac{ab^2c^3}{abc} =$
15. $\frac{m^5p^2x^4}{mp^2x^2} =$
16. $\frac{-51abdy^3}{3bdy} =$
17. $\frac{225m^2y}{25my^2} =$
18. $\frac{30x^2y^3}{-5x^3y} =$
19. $\frac{4a^2m^4x^5}{5a^5m^3x} =$
20. $\frac{42x^3y^2z^4}{7xy^2z^3} =$
21. $\frac{-3a^2b^3c^4d^5}{-a^4b^2cd^3} =$
22. $\frac{12am^5n^4p^3q^2}{4m^2n^3p^4q^5} =$
23. $(4a^2bz^3 \times 10a^2b^3z) \div 5a^3b^2z^2 =$
24. $(21x^2y^4z^6 \div 3xy^2z)(-2x^3y^2z) =$
25. $104ab^3x^9 \div (91a^5b^6x^7 \div 7a^4b^4x) =$
26. $(24a^5b^3x \div 3a^2b^2) + (35a^6b^2x^2 \div -5a^3bx) =$
27. $85a^{4m+1} \div 5a^{4m-2} =$
28. $84a^{n-4} \div 12a^2 =$

Of Polynomials by Monomials.

95. The product of $(a+b+c)\times p = ap+bp+cp$.

If the product of two factors be divided by one of the factors, the quotient is the other factor. Therefore,

$$(ap+bp+cp)\div p = a+b+c.$$

But a, b, and c are the quotients obtained by dividing each term, ap, bp, and cp, by p.

Therefore, to divide a polynomial by a monomial,

96. *Divide each term of the polynomial by the monomial.*

Ex. 22.

1. $(8ab-12ac)\div 4a = 2b-3c.$
2. $(15am-10bm+20cm)\div -5m = -3a+2b-4c.$
3. $(18amy-27bny+36cpy)\div -9y =$
4. $(21ax-18bx+15cx)\div -3x =$
5. $(12x^5-8x^3+4x)\div 4x =$
6. $(3x^3-6x^5+9x^7-12x^9)\div 3x^2 =$
7. $(35m^3y+28m^2y^2-14my^3)\div -7my =$
8. $(4a^4b-6a^3b^2+12a^2b^3)\div 2a^2b =$
9. $(12x^3y^3-15x^4y^2-24x^5y)\div -3x^2y =$
10. $(12x^5y^4-24x^4y^2+36x^3y^3-12x^2y^2)\div 12x^2y^2 =$
11. $(3a^4-2a^5b-a^6b^2)\div a^4 =$
12. $(3x^3yz^2+6x^4yz^3-15x^5y^2z^3+18x^6y^3z)\div -3x^3yz =$
13. $(-16a^3b^2c^5+8a^4b^2c^4-12a^5b^3c^3)\div -4a^2b^2c^2 =$

Of Polynomials by Polynomials.

97. If the divisor (one factor) $= a + b + c$,
and the quotient (other factor) $= n + p + q$,

$$\text{then the dividend (product)} = \begin{cases} an + bn + cn \\ + ap + bp + cp \\ + aq + bq + cq. \end{cases}$$

The first term of the dividend is an; that is, the product of a, the first term of the divisor, by n, the first term of the quotient. The first term n of the quotient is therefore found by dividing an, the first term of the dividend, by a, the first term of the divisor.

If the partial product formed by multiplying the entire divisor by n be subtracted from the dividend, the first term of the remainder ap is the product of a, the first term of the divisor, by p, the second term of the quotient. That is, the second term of the quotient is obtained by dividing the first term of the remainder by the first term of the divisor. In like manner, the third term of the quotient is obtained by dividing the first term of the new remainder by the first term of the divisor, and so on.

Therefore, to divide one polynomial by another,

98. *Divide the first term of the dividend by the first term of the divisor.*

Write the result as the first term of the quotient.

Multiply all the terms of the divisor by the first term of the quotient.

Subtract the product from the dividend.

If there be a remainder, consider it as a new dividend and proceed as before.

99. It is of great importance to *arrange both dividend and divisor according to the ascending or descending powers of some common letter, and to keep this order throughout the operation.*

Ex. 23.

Divide:

(1) $a^2+2ab+b^2$ by $a+b$;

$$\begin{array}{l|l}
a^2+2ab+b^2 & a+b \\ \hline
\underline{a^2+ab} & a+b \\
\quad ab+b^2 & \\
\quad \underline{ab+b^2} &
\end{array}$$

(2) a^2-b^2 by $a+b$;

$$\begin{array}{l|l}
a^2-b^2 & a+b \\ \hline
\underline{a^2+ab} & a-b \\
\quad -ab-b^2 & \\
\quad \underline{-ab-b^2} &
\end{array}$$

(3) $a^2-2ab+b^2$ by $a-b$;

$$\begin{array}{l|l}
a^2-2ab+b^2 & a-b \\ \hline
\underline{a^2-ab} & a-b \\
\quad -ab+b^2 & \\
\quad \underline{-ab+b^2} &
\end{array}$$

(4) $4a^4x^2-4a^2x^4+x^6-a^6$ by x^2-a^2;

$$\begin{array}{l|l}
x^6-4a^2x^4+4a^4x^2-a^6 & x^2-a^2 \\ \hline
\underline{x^6-a^2x^4} & x^4-3a^2x^2+a^4 \\
\quad -3a^2x^4+4a^4x^2-a^6 & \\
\quad \underline{-3a^2x^4+3a^4x^2} & \\
\qquad a^4x^2-a^6 & \\
\qquad \underline{a^4x^2-a^6} &
\end{array}$$

(5) $22a^2b^2+15b^4+3a^4-10a^3b-22ab^3$ by a^2+3b^2-2ab;

$$\begin{array}{l|l}
3a^4-10a^3b+22a^2b^2-22ab^3+15b^4 & a^2-2ab+3b^2 \\ \hline
\underline{3a^4-6a^3b+9a^2b^2} & 3a^2-4ab+5b^2 \\
\quad -4a^3b+13a^2b^2-22ab^3 & \\
\quad \underline{-4a^3b+8a^2b^2-12ab^3} & \\
\qquad 5a^2b^2-10ab^3+15b^4 & \\
\qquad \underline{5a^2b^2-10ab^3+15b^4} &
\end{array}$$

Divide:

6. $x^2-7x+12$ by $x-3$.

7. x^2+x-72 by $x+9$.

8. $2x^3-x^2+3x-9$ by $2x-3$.

9. $6x^3+14x^2-4x+24$ by $2x+6$.

10. $3x^2+x+9x^3-1$ by $3x-1$.

11. $7x^3+58x-24x^2-21$ by $7x-3$.

12. x^6-1 by $x-1$.

13. $a^3-2ab^2+b^3$ by $a-b$.

14. x^4-81y^4 by $x-3y$.

15. x^5-y^5 by $x-y$.

16. a^5+32b^5 by $a+2b$.

17. $2a^4+27ab^3-81b^4$ by $a+3b$.

18. $x^4+11x^2-12x-5x^3+6$ by $3+x^2-3x$.

19. $x^4-9x^2+x^3-16x-4$ by x^2+4+4x.

20. $36+x^4-13x^2$ by $6+x^2+5x$.

21. x^4+64 by x^2+4x+8.

22. $x^4+x^3+57-35x-24x^2$ by x^2-3+2x.

23. $1-x-3x^2-x^5$ by $1+2x+x^2$.

24. x^6-2x^3+1 by x^2-2x+1.

25. $a^4+2a^2b^2+9b^4$ by $a^2-2ab+3b^2$.

26. $4x^5-x^3+4x$ by $2+2x^2+3x$.

27. a^5-243 by $a-3$.

28. $18x^4+82x^2+40-67x-45x^3$ by $3x^2+5-4x$.

29. $x^4-6xy-9x^2-y^2$ by x^2+y+3x.

30. $x^4 + 9x^2y^2 - 6x^3y - 4y^4$ by $x^2 - 3xy + 2y^2$.

31. $x^4 + x^2y^2 + y^4$ by $x^2 - xy + y^2$.

32. $x^5 + x^3 + x^4y + y^3 - 2xy^2 - x^3y^2$ by $x^3 + x - y$.

33. $2x^2 - 3y^2 + xy - xz - 4yz - z^2$ by $2x + 3y + z$.

34. $12 + 82x^2 + 106x^4 - 70x^5 - 112x^3 - 38x$ by $3 - 5x + 7x^2$.

35. $x^5 + y^5$ by $x^4 - x^3y + x^2y^2 - xy^3 + y^4$.

36. $2x^4 + 2x^2y^2 - 2xy^3 - 7x^3y - y^4$ by $2x^2 + y^2 - xy$.

37. $16x^4 + 4x^2y^2 + y^4$ by $4x^2 - 2xy + y^2$.

38. $32a^5b + 8a^3b^3 - ab^5 - 4a^2b^4 - 56a^4b^2$ by $b^3 - 4a^2b + 6ab^2$.

39. $1 + 5x^3 - 6x^4$ by $1 - x + 3x^2$.

40. $1 - 52a^4b^4 - 51a^3b^3$ by $4a^2b^2 + 3ab - 1$.

41. $x^7y - xy^7$ by $x^3y + 2xy^3 - 2x^2y^2 - y^4$.

42. $x^6 + 15x^4y^2 + 15x^2y^4 + y^6 - 6x^5y - 6xy^5 - 20x^3y^3$ by $x^3 - 3x^2y + 3xy^2 - y^3$.

43. $a^7 + 2a^3b^4 - 2a^4b^3 - 2a^6b - 6a^2b^5 - 3ab^6$ by $a^3 - 2a^2b - ab^2$.

44. $81x^6y + 18x^2y^5 - 54x^5y^2 - 18x^3y^4 - 18xy^6 - 9y^7$ by $3x^4 + x^2y^2 + y^4$.

45. $a^4 + 2a^3b + 8a^2b^2 + 8ab^3 + 16b^4$ by $a^2 + 4b^2$.

46. $8y^6 - x^6 + 21x^3y^3 - 24xy^5$ by $3xy - x^2 - y^2$.

47. $16a^4 + 9b^4 + 8a^2b^2$ by $4a^2 + 3b^2 - 4ab$.

48. $a^3 + b^3 + c^3 - 3abc$ by $a + b + c$.

49. $a^3 + 8b^3 + c^3 - 6abc$ by $a^2 + 4b^2 + c^2 - ac - 2ab - 2bc$.

50. $a^3 + b^3 + c^3 + 3a^2b + 3ab^2$ by $a + b + c$.

100. There are some cases in Division which occur so often in algebraic operations that they should be carefully noticed and remembered.

CASE I.

The student may easily verify the following results:

(1) $\dfrac{a^3-b^3}{a-b}=a^2+ab+b^2.$

(2) $\dfrac{27a^3-8b^3}{3a-2b}=9a^2+6ab+4b^2.$

(3) $\dfrac{a^5-b^5}{a-b}=a^4+a^3b+a^2b^2+ab^3+b^4.$

(4) $\dfrac{a^5-32b^5}{a-2b}=a^4+2a^3b+4a^2b^2+8ab^3+16b^4.$

From these results it may be assumed that:

101. *The difference of two equal odd powers of any two numbers is divisible by the difference of the numbers.*

It will also be seen that:

I. The number of terms in the quotient is equal to the exponent of the powers.

II. The signs of the quotient are all positive.

III. The first term of the quotient is obtained, as usual, by dividing the first term of the dividend by the first term of the divisor.

IV. Each succeeding term of the quotient may be obtained by dividing the preceding term of the quotient by the first term of the divisor, and multiplying the result by the second term of the divisor (disregarding the sign).

Ex. 24.

Write by inspection the results in the following examples:

1. $(y^3-1)\div(y-1)$.
2. $(b^3-125)\div(b-5)$.
3. $(a^3-216)\div(a-6)$.
4. $(x^3-343)\div(x-7)$.
5. $(x^5-y^5)\div(x-y)$.
6. $(a^5-1)\div(a-1)$.
7. $(1-8x^3)\div(1-2x)$.
8. $(x^5-32b^5)\div(x-2b)$.

Case II.

(1) $\dfrac{a^3+b^3}{a+b}=a^2-ab+b^2.$

(2) $\dfrac{27x^3+8y^3}{3x+2y}=9x^2-6xy+4y^2.$

(3) $\dfrac{a^5+b^5}{a+b}=a^4-a^3b+a^2b^2-ab^3+b^4.$

(4) $\dfrac{243x^5+32y^5}{3x+2y}=81x^4-54x^3y+36x^2y^2-24xy^3+16y^4.$

From these results it may be assumed that:

102. *The sum of two equal odd powers of two numbers is divisible by the sum of the numbers.*

The quotient may be found as in Case I., but the signs are alternately plus and minus.

Ex. 25.

Write by inspection the results in the following examples:

1. $(x^3+y^3)\div(x+y)$.
2. $(x^5+y^5)\div(x+y)$.
3. $(1+8a^3)\div(1+2a)$.
4. $(27a^3+b^3)\div(3a+b)$.
5. $(8a^3x^3+1)\div(2ax+1)$.
6. $(x^3+27y^3)\div(x+3y)$.
7. $(a^5+32b^5)\div(a+2b)$.
8. $(512x^3y^3+z^3)\div(8xy+z)$.

CASE III.

(1) $\frac{x^2-y^2}{x-y}=x+y.$ (2) $\frac{x^4-y^4}{x-y}=x^3+x^2y+xy^2+y^3.$

(3) $\frac{x^2-y^2}{x+y}=x-y.$ (4) $\frac{x^4-y^4}{x+y}=x^3-x^2y+xy^2-y^3.$

From these results it may be assumed that:

103. *The difference of two equal even powers of two numbers is divisible by the difference and also by the sum of the numbers.*

When the divisor is the difference of the numbers, the quotient is found as in Case I.

When the divisor is the sum of the numbers, the quotient is found as in Case II.

Ex. 26.

Write by inspection the results in the following examples:

1. $(x^4-y^4)\div(x-y)$.
2. $(x^4-y^4)\div(x+y)$.
3. $(a^6-x^6)\div(a-x)$.
4. $(a^6-x^6)\div(a+x)$.
5. $(x^4-81y^4)\div(x-3y)$.
6. $(x^4-81y^4)\div(x+3y)$.
7. $(16x^4-1)\div(2x-1)$.
8. $(16x^4-1)\div(2x+1)$.
9. $(81a^4x^4-1)\div(3ax-1)$.
10. $(81a^4x^4-1)\div(3ax+1)$.

CASE IV.

It may be easily verified that:

104. *The sum of two equal even powers of two numbers is not divisible by either the sum or the difference of the numbers.*

But when the exponent of each of the two equal powers is composed of an *odd* and an *even* factor, the sum of the given powers is divisible by the sum of the powers expressed by the even factor.

Thus, x^6+y^6 is not divisible by $x+y$ or by $x-y$, but is divisible by x^2+y^2.

The quotient may be found as in Case II.

Ex. 27.

Write by inspection the results in the following examples:

1. $(x^6+y^6)\div(x^2+y^2)$.
2. $(a^6+1)\div(a^2+1)$.
3. $(a^{10}+y^{10})\div(a^2+y^2)$.
4. $(b^{10}+1)\div(b^2+1)$.
5. $(a^{12}+b^{12})\div(a^4+b^4)$.
6. $(x^{12}+1)\div(x^4+1)$.
7. $(64x^6+y^6)\div(4x^2+y^2)$.
8. $(64+a^6)\div(4+a^2)$.

NOTE. The introduction of negative numbers requires an extension of the meanings of some terms common to arithmetic and algebra. But every such extension of meaning must be consistent with the sense previously attached to the term and with general laws already established.

Addition in algebra does not necessarily imply *augmentation*, as it does in arithmetic. Thus, $7+(-5)=2$. The word **sum**, however, is used to denote the result.

Such a result is called the **algebraic sum**, when it is necessary to distinguish it from the *arithmetical sum*, which would be obtained by adding the *absolute values* of the numbers.

The general definition of Addition is, the operation of uniting two or more numbers in a *single expression* written in its simplest form.

The general definition of Subtraction is, the operation of finding from two given numbers, called *minuend* and *subtrahend*, a third number, called *difference*, which added to the subtrahend will give the minuend.

The general definition of Multiplication is, the operation of finding from two given numbers, called *multiplicand* and *multiplier*, a third number, called *product*, which may be formed from the multiplicand as the multiplier is formed from unity.

The general definition of Division is, the operation of finding the *other factor* when the *product* of two factors and *one factor* are given.

CHAPTER V.

SIMPLE EQUATIONS.

105. An **equation** is a statement that two expressions are equal. Thus, $4x - 12 = 8$.

106. Every equation consists of two parts, called the first and second *sides*, or *members*, of the equation.

107. An **identical equation** is one in which the two sides are equal, whatever numbers the letters stand for. Thus, $(x + b)(x - b) = x^2 - b^2$.

108. An **equation of condition** is one which is true only when the letters stand for particular values. Thus, $x + 5 = 8$ is true only when $x = 3$.

109. A letter to which a particular value must be given in order that the statement contained in an equation may be true is called an *unknown quantity.*

110. The *value* of the unknown quantity is the **number** which substituted for it will *satisfy* the equation, and is called a *root* of the equation.

111. To **solve** an equation is to find the value of the unknown quantity.

112. A **simple equation** is one which contains only the *first* power of the unknown quantity, and is also called an equation of the *first degree.*

113. *If equal changes be made in both sides of an equation, the results will be equal.* § 43.

(1) To find the value of x in $x+b=a$.

$$x+b=a;$$

Subtract b from each side, $x+b-b=a-b$;

Cancel $+b-b$, $x=a-b$.

(2) To find the value of x in $x-b=a$.

$$x-b=a;$$

Subtract $-b$ from each side, $x-b+b=a+b$;

Cancel $-b+b$, $x=a+b$.

The result in each case is the same as if b were transposed to the other side of the equation with its sign changed. Therefore,

114. *Any term may be transposed from one side of an equation to the other provided its sign be changed.*

For, in this transposition, the same number is subtracted from each side of the equation.

115. The signs of all the terms on each side of an equation may be changed; for, this is in effect transposing every term.

116. When the known and unknown quantities of an equation are connected by the sign + or —, they may be separated by transposing the known quantities to one side and the unknown to the other.

117. Hence, to solve an equation with one unknown quantity,

Transpose all the terms involving the unknown quantity to the left side, and all the other terms to the right side:

combine the like terms, and divide both sides by the coefficient of the unknown quantity.

118. To verify the result, substitute the value of the unknown quantity in the original equation.

Ex. 28.

Find the value of x in:

1. $5x-1=19$.
2. $3x+6=12$.
3. $24x=7x+34$.
4. $8x-29=26-3x$.
5. $12-5x=19-12x$.
6. $3x+6-2x=7x$.
7. $5x+50=4x+56$.
8. $16x-11=7x+70$.
9. $24x-49=19x-14$.
10. $3x+23=78-2x$.
11. $26-8x=80-14x$.
12. $13-3x=5x-3$.
13. $3x-22=7x+6$.
14. $8+4x=12x-16$.
15. $5x-(3x-7)=4x-(6x-35)$.
16. $6x-2(9-4x)+3(5x-7)=10x-(4+16x+35)$.
17. $9x-3(5x-6)+30=0$.
18. $x-7(4x-11)=14(x-5)-19(8-x)-61$.
19. $(x+7)(x-3)=(x-5)(x-15)$.
20. $(x-8)(x+12)=(x+1)(x-6)$.
21. $(x-2)(7-x)+(x-5)(x+3)-2(x-1)+12=0$.
22. $(2x-7)(x+5)=(9-2x)(4-x)+229$.
23. $14-x-5(x-3)(x+2)+(5-x)(4-5x)=45x-76$.
24. $(x+5)^2-(4-x)^2=21x$.
25. $5(x-2)^2+7(x-3)^2=(3x-7)(4x-19)+42$.

Ex. 29.

PROBLEMS.

1. Find a number such that when 12 is added to its double the sum shall be 28.

Let $x =$ the number.
Then $2x =$ its double,
and $2x + 12 =$ double the number increased by 12.
But $28 =$ double the number increased by 12.

$$\therefore 2x + 12 = 28$$
$$2x = 28 - 12$$
$$2x = 16$$
$$x = 8$$

2. A farmer had two flocks of sheep, each containing the same number. He sold 21 sheep from one flock and 70 from the other, and then found that he had left in one flock twice as many as in the other. How many had he in each?

Let $x =$ number of sheep in each flock.
Then $x - 21 =$ number of sheep left in one flock,
and $x - 70 =$ number of sheep left in the other.

$$\therefore x - 21 = 2(x - 70)$$
$$x - 21 = 2x - 140$$
$$x - 2x = -140 + 21$$
$$-x = -119$$
$$x = 119$$

3. A and B had equal sums of money; B gave A \$5, and then 3 times A's money was equal to 11 times B's money. What had each at first?

Let $x =$ number of dollars each had.
Then $x + 5 =$ number of dollars A had after receiving \$5 from B,
and $x - 5 =$ number of dollars B had after giving A \$5.

$$\therefore 3(x+5) = 11(x-5)$$
$$3x + 15 = 11x - 55$$
$$3x - 11x = -55 - 15$$
$$-8x = -70$$
$$x = 8\tfrac{3}{4}$$

Therefore, each had \$8.75.

4. Find a number whose treble exceeds 50 by as much as its double falls short of 40.

Let $x =$ the number.
Then $3x =$ its treble,
and $3x - 50 =$ the excess of its treble over 50;
also, $40 - 2x =$ the number its double lacks of 40.

$$\therefore 3x - 50 = 40 - 2x$$
$$3x + 2x = 40 + 50$$
$$5x = 90$$
$$x = 18$$

5. What two numbers are those whose difference is 14, and whose sum is 48?

Let $x =$ the larger number.
Then $48 - x =$ the smaller number,
and $x - (48 - x) =$ the difference of the numbers.
But $14 =$ the difference of the numbers.

$$\therefore x - (48 - x) = 14$$
$$x - 48 + x = 14$$
$$2x = 62$$
$$x = 31$$

Therefore, the two numbers are 31 and 17.

6. To the double of a certain number I add 14, and obtain as a result 154. What is the number?

7. By adding 46 to a certain number, I obtain as a result a number three times as large as the original number. Find the original number.

8. One number is three times as large as another. If I take the smaller from 16 and the greater from 30, the remainders are equal. What are the numbers?

9. Divide the number 92 into four parts, such that the first exceeds the second by 10, the third by 18, and the fourth by 24.

10. The sum of two numbers is 20; and if three times the smaller number be added to five times the greater, the sum is 84. What are the numbers?

11. The joint ages of a father and son are 80 years. If the age of the son were doubled, he would be 10 years older than his father. What is the age of each?

12. A man has 6 sons, each 4 years older than the next younger. The eldest is three times as old as the youngest. What is the age of each?

13. Add $24 to a certain sum, and the amount will be as much above $80 as the sum is below $80. What is the sum?

14. Thirty yards of cloth and 40 yards of silk together cost $330; and the silk cost twice as much a yard as the cloth. How much does each cost a yard?

15. Find the number whose double increased by 24 exceeds 80 by as much as the number itself is less than 100.

16. The sum of $500 is divided among A, B, C, and D. A and B have together $280, A and C $260, and A and D $220. How much does each receive?

17. In a company of 266 persons composed of men, women, and children, there are twice as many men as women, and twice as many women as children. How many are there of each?

18. Find two numbers differing by 8, such that four times the less may exceed twice the greater by 10.

19. A is 58 years older than B, and A's age is as much above 60 as B's age is below 50. Find the age of each.

20. A has \$72 and B has \$52. B gives A a certain sum; then A has three times as much as B. How much did A receive from B?

21. Divide 90 into two such parts that four times one part may be equal to five times the other.

22. Divide 60 into two such parts that one part exceeds the other by 24.

23. Divide 84 into two such parts that one part may be less than the other by 36.

NOTE I. When we have to compare the ages of two persons at a given time, and also a number of years after or before the given time, we must remember that *both* persons will be so many years older or younger.

Thus, if x represent A's age, and $2x$ B's age, at the present time, A's age five years ago will be represented by $x-5$; and B's by $2x-5$. A's age five years hence will be represented by $x+5$; and B's age by $2x+5$.

24. A is twice as old as B, and 22 years ago he was three times as old as B. What is A's age?

25. A father is 30 and his son 6 years old. In how many years will the father be just twice as old as the son?

26. A is twice as old as B, and 20 years since he was three times as old. What is B's age?

27. A is three times as old as B, and 19 years hence he will be only twice as old as B. What is the age of each?

NOTE II. In problems involving quantities of the same kind expressed in different units, we must be careful to reduce all the quantities to the *same unit*.

Thus, if x denote a number of inches, all the quantities of the same kind involved in the problem must be reduced to inches.

28. A sum of money consists of dollars and twenty-five-cent pieces, and amounts to \$20. The number of coins is 50. How many are there of each sort?

29. A person bought 30 pounds of sugar of two different kinds, and paid for the whole \$2.94. The better kind cost 10 cents a pound and the poorer kind 7 cents a pound. How many pounds were there of each kind?

30. A workman was hired for 40 days, at \$1 for every day he worked, but with the condition that for every day he did not work he was to pay 45 cents for his board. At the end of the time he received \$22.60. How many days did he work?

31. A gentleman gave some children 10 cents each, and had a dollar left. He found that he would have required one dollar more to enable him to give them 15 cents each. How many children were there?

32. Two casks contain equal quantities of vinegar; from the first cask 34 quarts are drawn, from the second, 20 gallons; the quantity remaining in one vessel is now twice that in the other. How much did each cask contain at first?

33. A man has three times as many quarters as half-dollars, four times as many dimes as quarters, and twice as many half-dimes as dimes. The whole sum is \$7.30. How many coins has he altogether?

CHAPTER VI.

FACTORS.

119. In multiplication we determine the *product* of two given factors; it is often important to determine the *factors* of a given product.

120. CASE I. The simplest case is that in which all the terms of an expression have one common factor. Thus,

(1) $x^2 + xy = x(x + y)$.

(2) $6a^3 + 4a^2 + 8a = 2a(3a^2 + 2a + 4)$.

(3) $18a^3b - 27a^2b^2 + 36ab = 9ab(2a^2 - 3ab + 4)$.

Ex. 30.

Resolve into factors:

1. $5a^2 - 15a$.
2. $6a^3 + 18a^2 - 12a$.
3. $49x^2 - 21x + 14$.
4. $4x^3y - 12x^2y^2 + 8xy^3$.
5. $y^4 - ay^3 + by^2 + cy$.
6. $6a^5b^3 - 21a^4b^2 + 27a^3b^4$.
7. $54x^2y^6 + 108x^4y^8 - 243x^6y^9$.
8. $45x^7y^{10} - 90x^5y^7 - 360x^4y^8$.
9. $70a^3y^4 - 140a^2y^5 + 210ay^6$.
10. $32a^3b^6 + 96a^6b^8 - 128a^8b^9$.

121. Case II. Frequently the terms of an expression can be so arranged as to show a common factor. Thus,

$$(1)\quad x^2+ax+bx+ab=(x^2+ax)+(bx+ab)$$
$$=x(x+a)+b(x+a)$$
$$=(x+b)(x+a).$$

$$(2)\quad ac-ad-bc+bd=(ac-ad)-(bc-bd)$$
$$=a(c-d)-b(c-d)$$
$$=(a-b)(c-d).$$

Ex. 31.

Resolve into factors:

1. $x^2-ax-bx+ab.$
2. $ab+ay-by-y^2.$
3. $bc+bx-cx-x^2.$
4. $mx+mn+ax+an.$
5. $cdx^2-cxy+dxy-y^2.$
6. $abx-aby+pqx-pqy.$
7. $cdx^2+adxy-bcxy-aby^2.$
8. $abcy-b^2dy-acdx+bd^2x.$
9. $ax-ay-bx+by.$
10. $cdz^2-cyz+dyz-y^2.$

122. The square root of a number is one of the *two equal* factors of that number. Thus, the square root of 25 is 5; for $25=5\times 5$.

The square root of a^4 is a^2; for $a^4=a^2\times a^2$.

The square root of $a^2b^2c^2$ is abc; for $a^2b^2c^2=abc\times abc$.

In general, the square root of a power of a number is expressed by writing the number with an exponent equal to one-half the exponent of the power.

The square root of a product may be found by taking the square root of each factor, and finding the product of the roots.

The square root of a positive number may be either positive or negative; for,

$$a^2 = a \times a,$$

or

$$a^2 = -a \times -a;$$

but throughout this chapter only the positive value of the square root will be taken.

123. CASE III. From § 73 it is seen that a trinomial is often the product of two binomials. Conversely, a trinomial may, in certain cases, be resolved into two binomial factors. Thus,

To find the factors of

$$x^2 + 7x + 12.$$

The first term of each binomial factor will obviously be x.

The second terms of the two binomial factors must be two numbers

whose *product* is 12,

and whose *sum* is 7.

The only two numbers whose product is 12 and whose sum is 7 are 4 and 3.

$$\therefore x^2 + 7x + 12 = (x + 4)(x + 3).$$

Again, to find the factors of $x^2 + 5xy + 6y^2$.

The first term of each binomial factor will obviously be x.

The second terms of the two binomial factors must be two numbers

whose *product* is $6y^2$,

and whose *sum* is $5y$.

The only two numbers whose product is $6y^2$ and whose sum is $5y$ are $3y$ and $2y$.

$$\therefore x^2 + 5xy + 6y^2 = (x + 3y)(x + 2y).$$

Ex. 32.

Find the factors of:

1. $x^2+11x+24$.
2. $x^2+11x+30$.
3. $y^2+17y+60$.
4. $z^2+13z+12$.
5. $x^2+21x+110$.
6. $y^2+35y+300$.
7. $b^2+23b+102$.
8. x^2+3x+2.
9. x^2+7x+6.
10. $a^2+9ab+8b^2$.

124. CASE IV. To find the factors of

$$x^2-9x+20.$$

The second terms of the two binomial factors must be two numbers

whose *product* is 20,

and whose *sum* is -9.

The only two numbers whose product is 20 and whose sum is -9 are -5 and -4.

$$\therefore x^2-9x+20=(x-5)(x-4).$$

Ex. 33.

Resolve into factors:

1. $x^2-7x+10$.
2. $x^2-29x+190$.
3. $a^2-23a+132$.
4. $b^2-30b+200$.
5. $z^2-43z+460$.
6. x^2-7x+6.
7. $x^4-4a^2x^2+3a^4$.
8. $x^2-8x+12$.
9. $z^2-57z+56$.
10. y^6-7y^3+12.

125. CASE V. To find the factors of

$$x^2 + 2x - 3.$$

The second terms of the two binomial factors must be two numbers

whose *product* is -3,

and whose *sum* is $+2$.

The only two numbers whose product is -3 and whose sum is $+2$ are $+3$ and -1.

$$\therefore x^2 + 2x - 3 = (x+3)(x-1).$$

Ex. 34.

Resolve into factors:

1. $x^2 + 6x - 7$.
2. $x^2 + 5x - 84$.
3. $y^2 + 7y - 60$.
4. $y^2 + 12y - 45$.
5. $z^2 + 11z - 12$.
6. $z^2 + 13z - 140$.
7. $a^2 + 13a - 300$.
8. $a^2 + 25a - 150$.
9. $b^8 + 3b^4 - 4$.
10. $b^2c^2 + 3bc - 154$.

126. CASE VI. To find the factors of

$$x^2 - 5x - 66.$$

The second terms of the two binomial factors must be two numbers

whose *product* is -66,

and whose *sum* is -5.

The only two numbers whose product is -66 and whose sum is -5 are -11 and $+6$.

$$\therefore x^2 - 5x - 66 = (x-11)(x+6).$$

Ex. 35.

Resolve into factors:

1. $x^2 - 3x - 28.$
2. $y^2 - 7y - 18.$
3. $x^2 - 9x - 36.$
4. $z^2 - 11z - 60.$
5. $z^2 - 13z - 14.$
6. $a^2 - 15a - 100.$
7. $c^{10} - 9c^5 - 10.$
8. $x^2 - 8x - 20.$
9. $y^2 - 5ay - 50a^2.$
10. $a^2b^2 - 3ab - 4.$

We now proceed to the consideration of trinomials which are perfect squares. These are only particular forms of Cases III. and IV., but from their importance demand special attention.

127. Case VII. To find the factors of

$$x^2 + 18x + 81.$$

The second terms of the two binomial factors must be two numbers

whose *product* is 81,

and whose *sum* is 18.

The only two numbers whose product is 81 and whose sum is 18 are 9 and 9.

$$\therefore x^2 + 18x + 81 = (x + 9)(x + 9) = (x + 9)^2.$$

Ex. 36.

Resolve into factors:

1. $x^2 + 12x + 36.$
2. $x^2 + 28x + 196.$
3. $x^2 + 34x + 289.$
4. $z^2 + 2z + 1.$

5. $y^2 + 200y + 10{,}000$.
6. $z^4 + 14z^2 + 49$.
7. $x^2 + 36xy + 324y^2$.
8. $y^4 + 16y^2z^2 + 64z^4$.
9. $y^6 + 24y^3 + 144$.
10. $4a^2 + 12ab^2 + 9b^4$.

128. CASE VIII. To find the factors of

$$x^2 - 18x + 81.$$

The second terms of the two trinomials must be two numbers

whose *product* is 81,

and whose *sum* is -18.

The only two numbers whose product is 81 and whose sum is -18 are -9 and -9.

$$\therefore x^2 - 18x + 81 = (x-9)(x-9) = (x-9)^2.$$

Ex. 37.

1. $a^2 - 8a + 16$.
2. $a^2 - 30a + 225$.
3. $x^2 - 38x + 361$.
4. $x^2 - 40x + 400$.
5. $y^2 - 100y + 2500$.
6. $y^4 - 20y^2 + 100$.
7. $y^2 - 50yz + 625z^2$.
8. $x^4 - 32x^2y^2 + 256y^4$.
9. $z^6 - 34z^3 + 289$.
10. $4x^4y^2 - 20x^2y^3z + 25y^4z^2$.

129. CASE IX. An expression in the form of two squares, with the negative sign between them, is the product of two factors which may be determined as follows:

Take the square root of the first number, and the square root of the second number.

The *sum* of these roots will form the first factor;

The *difference* of these roots will form the second factor. Thus:

(1) $a^2 - b^2 = (a+b)(a-b)$.

(2) $a^2 - (b-c)^2 = \{a+(b-c)\}\{a-(b-c)\}$
$= \{a+b-c\}\{a-b+c\}$.

(3) $(a-b)^2-(c-d)^2=\{(a-b)+(c-d)\}\{(a-b)-(c-d)\}$
$=\{a-b+c-d\}\{a-b-c+d\}$.

130. The terms of an expression may often be arranged so as to form two squares with the negative sign between them, and the expression can then be resolved into factors. Thus:

$$\begin{aligned}
&a^2+b^2-c^2-d^2+2ab+2cd\\
&\quad=a^2+2ab+b^2-c^2+2cd-d^2\\
&\quad=(a^2+2ab+b^2)-(c^2-2cd+d^2)\\
&\quad=(a+b)^2-(c-d)^2\\
&\quad=\{(a+b)+(c-d)\}\{(a+b)-(c-d)\}\\
&\quad=\{a+b+c-d\}\{a+b-c+d\}.
\end{aligned}$$

131. An expression may often be resolved into three or more factors. Thus:

(1) $x^{16}-y^{16}=(x^8+y^8)(x^8-y^8)$
$=(x^8+y^8)(x^4+y^4)(x^4-y^4)$
$=(x^8+y^8)(x^4+y^4)(x^2+y^2)(x^2-y^2)$
$=(x^8+y^8)(x^4+y^4)(x^2+y^2)(x+y)(x-y)$.

(2) $4(ab+cd)^2-(a^2+b^2-c^2-d^2)^2$
$=\{2(ab+cd)+(a^2+b^2-c^2-d^2)\}$
$\{2(ab+cd)-(a^2+b^2-c^2-d^2)\}$
$=\{2ab+2cd+a^2+b^2-c^2-d^2\}$
$\{2ab+2cd-a^2-b^2+c^2+d^2\}$
$=\{(a^2+2ab+b^2)-(c^2-2cd+d^2)\}$
$\{(c^2+2cd+d^2)-(a^2-2ab+b^2)\}$

$$= \{(a+b)^2-(c-d)^2\}\{(c+d)^2-(a-b)^2\}$$
$$= \{a+b+(c-d)\}\{a+b-(c-d)\}$$
$$\{c+d+(a-b)\}\{c+d-(a-b)\}$$
$$= \{a+b+c-d\}\{a+b-c+d\}$$
$$\{c+d+a-b\}\{c+d-a+b\}.$$

Ex. 38.

Resolve into factors:

1. a^2-b^2.
2. a^2-16.
3. $4a^2-25$.
4. a^4-b^4.
5. a^4-1.
6. a^8-b^8.
7. a^8-1.
8. $36x^2-49y^2$.
9. $100x^2y^2-121a^2b^2$.
10. $1-49x^2$.
11. a^4-25b^2.
12. $(a-b)^2-c^2$.
13. $x^2-(a-b)^2$.
14. $(a+b)^2-(c+d)^2$.
15. $(x+y)^2-(x-y)^2$.
16. $2ab-a^2-b^2+1$.
17. $x^2-2yz-y^2-z^2$.
18. $x^2-2xy+y^2-z^2$.
19. $a^2+12bc-4b^2-9c^2$.
20. $a^2-2ay+y^2-x^2-2xz-z^2$.
21. $2xy-x^2-y^2+z^2$.
22. $x^2+y^2-z^2-d^2-2xy-2dz$.
23. $x^2-y^2+z^2-a^2-2xz+2ay$.
24. $2ab+a^2+b^2-c^2$.
25. $2xy-x^2-y^2+a^2+b^2-2ab$.
26. $(ax+by)^2-1$.
27. $1-x^2-y^2+2xy$.
28. $a^2-2ab+b^2-x^2$.
29. $a^2-b^2-2bc-c^2$.
30. $4x^4-9x^2+6x-1$.

132. Case X.

Since $$\frac{x^3-y^3}{x-y}=x^2+xy+y^2,$$

and $$\frac{x^5-y^5}{x-y}=x^4+x^3y+x^2y^2+xy^3+y^4,$$

and so on, it follows that the difference between two equal odd powers of two numbers is divisible by the difference between the numbers.

Ex. 39.

Resolve into factors:

1. $a^3 - b^3$.
2. $x^3 - 8$.
3. $x^3 - 343$.
4. $y^3 - 125$.
5. $y^3 - 216$.
6. $8x^3 - 27y^3$.
7. $64y^3 - 1000z^3$.
8. $729x^3 - 512y^3$.
9. $27a^3 - 1728$.
10. $1000a^3 - 1331b^3$.

133. Case XI.

Since $$\frac{x^3 + a^3}{x + a} = x^2 - ax + a^2,$$

and $$\frac{x^5 + y^1}{x + y} = x^4 - x^3y + x^2y^2 - xy^3 + y^4,$$

and so on, it follows that the sum of two equal odd powers of two numbers is divisible by the sum of the numbers.

Ex. 40.

Resolve into factors:

1. $x^3 + y^3$.
2. $x^3 + 8$.
3. $x^3 + 216$.
4. $y^3 + 64z^3$.
5. $64b^3 + 125c^3$.
6. $216a^3 + 512c^3$.
7. $729x^3 + 1728y^3$.
8. $x^5 + y^5$.
9. $x^7 + y^7$.
10. $32b^5 + 243c^5$.

134. CASE XII. The sum of *any two powers* of two numbers, whose exponents contain the *same odd* factor, is divisible by the sum of the powers obtained by dividing the exponents of the given powers by this odd factor.

Thus:

$$\frac{x^6 + y^6}{x^2 + y^2} = x^4 - x^2y^2 + y^4;$$

$$\frac{x^6 + y^9}{x^2 + y^3} = x^4 - x^2y^3 + y^6.$$

In like manner, $x^{10} + 32y^5$, which is equal to $x^{10} + (2y)^5$, is divisible by $x^2 + 2y$; but $x^4 + y^4$, whose exponents do not contain *an odd* factor, and $x^6 + y^{10}$, whose exponents do not contain the *same odd* factor, cannot be resolved into factors.

Ex. 41.

Resolve into factors:

1. $a^6 + b^6$.	3. $x^{12} + y^{12}$.	5. $x^6 + 1$.	7. $64a^6 + x^3$.
2. $a^{10} + 32b^5$.	4. $b^6 + 64c^6$.	6. $a^{12} + 1$.	8. $729 + c^6$.

135. CASE XIII. The factors, if any exist, of a polynomial of more than three terms can often be found by the application of principles already explained. Thus it is seen at a glance that the expression

$$a^3 - 3a^2b + 3ab^2 - b^3$$

fulfils, both in respect to exponents and coefficients, the laws stated in § 83 for writing the power of a binomial; and it is known at once that

$$a^3 - 3a^2b + 3ab^2 - b^3 = (a - b)^3.$$

Again, it is seen that the expression

$$x^2 - 2xy + y^2 + 2xz - 2yz + z^2$$

consists of *three* squares and *three* double products, and, from § 79, is the square of a *trinomial* which has for terms x, y, z.

It is also seen from the double product $-2xy$, that x and y have *unlike* signs;

and from the double product $2xz$, that x and z have *like* signs. Hence,

$$x^2 - 2xy + y^2 + 2xz - 2yz + z^2 = (x - y + z)^2.$$

Ex. 42.

Resolve into factors:

1. $a^3 + 3a^2b + 3ab^2 + b^3$.
2. $a^3 + 3a^2 + 3a + 1$.
3. $a^3 - 3a^2 + 3a - 1$.
4. $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$.
5. $x^4 - 4x^3 + 6x^2 - 4x + 1$.
6. $a^4 - 4a^3c + 6a^2c^2 - 4ac^3 + c^4$.
7. $x^2 + 2xy + y^2 + 2xz + 2yz + z^2$.
8. $x^2 - 2xy + y^2 - 2xz + 2yz + z^2$.
9. $a^2 + b^2 + c^2 + 2ab - 2ac - 2bc$.

Ex. 43.

MISCELLANEOUS EXAMPLES.

The following expressions are to be resolved into factors by the principles already explained. The student should first carefully remove all monomial factors from the expressions.

1. $5x^2 - 15x - 20$.
2. $2x^5 - 16x^4 + 24x^3$.
3. $3a^2b^2 - 9ab - 12$.
4. $a^2 + 2ax + x^2 + 4a + 4x$.
5. $a^2 - 2ab + b^2 - c^2$.
6. $x^2 - 2xy + y^2 - c^2 + 2cd - d^2$.

7. a^2-b^2-a-b.

8. $x^2-y^2-xz+yz$.

9. $ab-ac-b^2+bc$.

10. $3x^2-3xz-xy+yz$.

11. $a^2-x^2-ab-bx$.

12. $a^2-2ax+x^2+a-x$.

13. $3x^2-3y^2-2x+2y$.

14. $x^4+x^3+x^2+x$.

15. $a^4x^4-a^3x^3-a^2x^2+1$.

16. x^6-y^6.

17. x^6+y^6.

18. $x^{12}y+y^{13}$.

19. a^4c-c^5.

20. $x^2+4x-21$.

21. $3a^2-21ab+30b^2$.

22. $4a^2-4ab+b^2$.

23. $16x^2-80xy+100y^2$.

24. $36a^2x^3y^2-25b^2x^5y^2$.

25. $9x^2y^4-30xy^2z+25z^2$.

26. $16x^5-x$.

27. $x^2-2xy-2xz+y^2+2yz+z^2$.

28. $1-x+x^2-x^3$.

29. $x^2+20x+91$.

30. $x^2-5x-24$.

31. $(x^2-y^2-z^2)^2-4y^2z^2$.

32. $3x^3-x^2+3x-1$.

33. $x^2-2mx+m^2-n^2$.

34. $4a^2b^2-(a^2+b^2-c^2)^2$.

35. a^7+a^5.

36. $1-14a^3x+49a^6x^2$.

37. $y^2-4y-117$.

38. $x^2+6x-135$.

39. $4a^2-12ab+9b^2-4c^2$.

40. $a^3-b^3-3ab(a-b)$.

41. $x^3+y^3+3xy(x+y)$.

42. $m^2p-m^2q-n^2p+n^2q$.

43. $2x^3+4x^2-70x$.

44. $16a^3x-2x^4$.

45. $32bx^3-4by^3$.

46. $x-27x^4$.

47. $x^{12}-y^{12}$.

48. $49m^2-121n^2$.

49. $16-81y^4$.

50. x^3-x^2+x-1.

51. $x^2+2x+1-y^2$.

52. $x^2-53x+360$.

53. $125x^5+350x^3y^2+245xy^4$.

54. $a^2-2ad+d^2-4b^2+12bc-9c^2$.

CHAPTER VII.

Common Factors and Multiples.

136. A **common factor** of two or more expressions is an expression which is contained in each of them without a remainder. Thus,

$5a$ is a common factor of $20a$ and $25a$;

$3x^2y^2$ is a common factor of $12x^2y^2$ and $15x^3y^3$.

137. Two expressions which have no common factor except 1, are said to be *prime* to each other.

138. The **Highest Common Factor** of two or more expressions is the product of all the factors common to the expressions.

Thus, $3a^2$ is the highest common factor of $3a^2$, $6a^3$, and $12a^4$.

$5x^2y^2$ is the highest common factor of $10x^3y^2$ and $15x^2y^3$.

For brevity, H. C. F. will be used for Highest Common Factor.

(1) Find the H. C. F. of $42a^3b^2x$ and $21a^2b^3x^2$.

$$42a^3b^2x = 2 \times 3 \times 7 \times a^3 \times b^2 \times x;$$
$$21a^2b^3x^2 = 3 \times 7 \times a^2 \times b^3 \times x^2.$$
$$\therefore \text{ the H. C. F.} = 3 \times 7 \times a^2 \times b^2 \times x = 21a^2b^2x.$$

(2) Find the H. C. F. of $2a^2x + 2ax^2$ and $3abxy + 3bx^2y$.

$$2a^2x + 2ax^2 = 2ax(a + x);$$
$$3abxy + 3bx^2y = 3bxy(a + x).$$
$$\therefore \text{ the H. C. F.} = x(a + x).$$

(3) Find the H. C. F. of

$$8a^2x^2 - 24a^2x + 16a^2 \text{ and } 12ax^2y - 12axy - 24ay.$$

$$\begin{aligned} 8a^2x^2 - 24a^2x + 16a^2 &= 8a^2(x^2 - 3x + 2) \\ &= 2^3a^2(x-1)(x-2); \\ 12ax^2y - 12axy - 24ay &= 12ay(x^2 - x - 2) \\ &= 2^2 \times 3ay(x+1)(x-2). \\ \therefore \text{the H. C. F.} &= 2^2a(x-2) \\ &= 4a(x-2). \end{aligned}$$

Hence, to find the H. F. C. of two or more expressions:

Resolve each expression into its lowest factors.

Select from these the lowest power of each common factor, and find the product of these powers.

Ex. 44.

Find the H. C. F. of:

1. $18ab^2c^3d$ and $36a^3bcd^2$.
2. $17pq^2$, $34p^2q$, and $51p^3q^3$.
3. $8x^2y^3z^4$, $12x^3y^2z^3$, and $20x^4y^3z^2$.
4. $30x^4y^5$, $90x^2y^3$, and $120x^3y^4$.
5. $a^2 - b^2$ and $a^3 - b^3$.
6. $a^2 - x^2$ and $(a - x)^2$.
7. $a^3 + x^3$ and $(a + x)^3$.
8. $9x^2 - 1$ and $(3x + 1)^2$.
9. $7x^2 - 4x$ and $7a^2x - 4a^2$.
10. $12a^3x^2y - 4a^3xy^2$ and $30a^2x^3y^2 - 10a^2x^2y^3$.
11. $8a^3b^2c - 12a^2bc^3$ and $6ab^4c + 4ab^3c^2$.
12. $x^2 - 2x - 3$ and $x^2 + x - 12$.
13. $2a^3 - 2ab^2$ and $4b(a + b)^2$.
14. $12x^3y(x - y)(x - 3y)$ and $18x^2(x - y)(3x - y)$.
15. $3x^3 + 6x^2 - 24x$ and $6x^3 - 96x$.

16. $ac\,(a-b)(a-c)$ and $bc\,(b-a)(b-c)$.
17. $10x^3y - 60x^2y^2 + 5xy^3$ and $5x^2y^2 - 5xy^3 - 100y^4$.
18. $x\,(x+1)^2$, $x^2\,(x^2-1)$, and $2x\,(x^2-x-2)$.
19. $3x^2-6x+3$, $6x^2+6x-12$, and $12x^2-12$.
20. $6\,(a-b)^4$, $8\,(a^2-b^2)^2$, and $10\,(a^4-b^4)$.
21. x^2-y^2, $(x+y)^2$, and $x^2+3xy+2y^2$.
22. x^2-y^2, x^3-y^3, and $x^2-7xy+6y^2$.
23. x^2-1, x^3-1, and x^2+x-2.

139. When it is required to find the H. C. F. of two or more expressions which cannot readily be resolved into their factors, the method to be employed is similar to that of the corresponding case in arithmetic. And as that method consists in obtaining pairs of continually decreasing numbers which contain as a factor the H. C. F. required; so in algebra, pairs of expressions of continually decreasing degrees are obtained, which contain as a factor the H. C. F. required.

140. By this method, find the H. C. F. of

$$2x^2+x-3 \text{ and } 4x^3+8x^2-x-6.$$

$$\begin{array}{l}
2x^2+x-3)\,4x^3+8x^2-x-6\,(2x+3 \\
\qquad\qquad\quad \underline{4x^3+2x^2-6x} \\
\qquad\qquad\qquad\quad 6x^2+5x-6 \\
\qquad\qquad\qquad\quad \underline{6x^2+3x-9} \\
\qquad\qquad\qquad\qquad 2x+3)\,2x^2+x-3\,(x-1 \\
\qquad\qquad\qquad\qquad\qquad\quad \underline{2x^2+3x} \\
\qquad\qquad\qquad\qquad\qquad\qquad -2x-3 \\
\qquad\qquad\qquad\qquad\qquad\qquad \underline{-2x-3}
\end{array}$$

$\therefore$ the H. C. F. $= 2x+3$.

The given expressions are arranged according to the descending powers of x.

The expression whose first term is of the lower degree is taken for the divisor; and each division is continued until the first term of the remainder is of lower degree than that of the divisor.

141. This method is of use only to determine the **compound** factor of the H. C. F. **Simple** factors of the given expressions must first be separated from them, and the highest common factor of these must be reserved to be multiplied into the compound factor obtained.

Find the H. C. F. of

$$12x^4+30x^3-72x^2 \text{ and } 32x^3+84x^2-176x.$$

$$12x^4+30x^3-72x^2=6x^2(2x^2+5x-12).$$

$$32x^3+84x^2-176x=4x\,(8x^2+21x-44).$$

$6x^2$ and $4x$ have $2x$ *common.*

$$
\begin{array}{rl}
2x^2+5x-12) & 8x^2+21x-44\,(4 \\
 & \underline{8x^2+20x-48} \\
 & \qquad x+4)\,2x^2+5x-12\,(2x-3 \\
 & \qquad\qquad \underline{2x^2+8x\qquad} \\
 & \qquad\qquad\quad -3x-12 \\
 & \qquad\qquad\quad \underline{-3x-12}
\end{array}
$$

$\therefore$ the H. C. F. $=2x\,(x+4)$.

142. Modifications of this method are sometimes needed.

(1) Find the H. C. F. of $4x^2-8x-5$ and $12x^2-4x-65$.

$$
\begin{array}{rl}
4x^2-8x-5) & 12x^2-4x-65\,(3 \\
 & \underline{12x^2-24x-15} \\
 & \qquad 20x-50
\end{array}
$$

The first division ends here, for $20x$ is of lower degree than $4x^2$. But if $20x-50$ be made the divisor, $4x^2$ will not contain $20x$ an *integral* number of times.

Now, it is to be remembered that the H. C. F. sought *is contained in the remainder* $20x-50$, and that it is a *compound factor.* Hence if the *simple factor* 10 be removed, the H. C. F. must still be con-

tained in $2x-5$, and therefore the process may be continued with $2x-5$ for a divisor.

$$\begin{array}{r}
2x-5)\,4x^2-8x-5\,(2x+1\\
\underline{4x^2-10x}\\
2x-5\\
\underline{2x-5}\\
\end{array}$$

$\therefore$ the H. C. F. $=2x-5$.

(2) Find the H. C. F. of

$21x^3-4x^2-15x-2$ and $21x^3-32x^2-54x-7$.

$$\begin{array}{r}
21x^3-4x^2-15x-2)\,21x^3-32x^2-54x-7\,(1\\
\underline{21x^3-4x^2-15x-2}\\
-28x^2-39x-5
\end{array}$$

The difficulty here cannot be obviated by *removing* a simple factor from the remainder, for $-28x^2-39x-5$ has no simple factor. In this case, the expression $21x^3-4x^2-15x-2$ must be *multiplied* by the simple factor 4 to make its first term divisible by $-28x^2$.

The *introduction* of such a factor can in no way affect the H. C. F. sought; for the H. C. F. contains only factors *common to the remainder and the last divisor*, and 4 is not a factor of the remainder.

The *signs* of all the terms of the remainder may be changed; for if an expression A is divisible by $-F$, it is divisible by $+F$.

The process then is continued by changing the signs of the remainder and multiplying the divisor by 4.

$$\begin{array}{lr}
 & 28x^2+39x+5)\,84x^3-16x^2-60x-8\,(3x\\
 & \underline{84x^3+117x^2+15x}\\
 & -133x^2-75x-8\\
\text{Multiply by } -4, & \underline{-4}\\
 & 532x^2+300x+32\,(19\\
 & \underline{532x^2+741x+95}\\
\text{Divide by } -63, & -63)\,\underline{-441x-63}\\
 & 7x+1
\end{array}$$

$$\begin{array}{r}
7x+1)\,28x^2+39x+5\,(4x+5 \\
\underline{28x^2+4x} \\
35x+5 \\
\underline{35x+5}
\end{array}$$

$\therefore$ the H. C. F. $= 7x+1.$

(3) Find the H. C. F. of

$$8x^2+2x-3 \text{ and } 6x^3+5x^2-2.$$

$$\begin{array}{rl}
 & 6x^3+5x^2-2 \\
 & 4 \\
8x^2+2x-3) & \overline{24x^3+20x^2-8}\;(3x+7 \\
 & \underline{24x^3+6x^2-9x} \\
 & 14x^2+9x-8 \\
\text{Multiply by 4,} & \underline{4} \\
 & 56x^2+36x-32 \\
 & \underline{56x^2+14x-21} \\
\text{Divide by 11,} & 11)\underline{22x-11} \\
 & 2x-1)\,8x^2+2x-3\,(4x+3 \\
 & \underline{8x^2-4x} \\
 & 6x-3 \\
 & \underline{6x-3}
\end{array}$$

$\therefore$ the H. C. F. $= 2x-1.$

In this case it is necessary to multiply by 4 the *given expression* $6x^3+5x^2-2$ to make its first term divisible by $8x^2$, 4 being obviously not a *common* factor.

The following arrangement of the work will be found most convenient:

$$\begin{array}{r|r|l}
8x^2+2x-3 & 6x^3+5x^2-2 & \\
\underline{8x^2-4x} & \underline{4} & \\
6x-3 & 24x^3+20x^2-8 & 3x \\
\underline{6x-3} & \underline{24x^3+6x^2-9x} & \\
 & 14x^2+9x-8 & \\
 & \underline{4} & \\
 & 56x^2+36x-32 & +7 \\
 & \underline{56x^2+14x-21} & \\
 & 11)\underline{22x-11} & \\
 & 2x-1 & 4x+3
\end{array}$$

143. From the foregoing examples it will be seen that, in the algebraic process of finding the highest common factor, the following steps, in the order here given, must be carefully observed:

I. Simple factors of the given expressions are to be removed from them, and the highest common factor of these is to be reserved as a factor of the H. C. F. sought.

II. The resulting compound expressions are to be arranged according to the *descending* powers of a common letter; and that expression which is of the lower degree is to be taken for the divisor; or, if both are of the same degree, that whose first term has the smaller coefficient.

III. Each division is to be continued until the remainder is of lower degree than the divisor.

IV. If the final remainder of any division is found to contain a factor that is not a *common* factor of the given expressions, *this factor is to be removed;* and the resulting expression is to be used as the next divisor.

V. A dividend whose first term is not exactly divisible by the first term of the divisor, is to be *multiplied* by such an expression as will make it thus divisible.

Ex. 45.

Find the H. C. F. of:

1. $5x^2+4x-1,\ 20x^2+21x-5.$
2. $2x^3-4x^2-13x-7,\ 6x^3-11x^2-37x-20.$
3. $6a^4+25a^3-21a^2+4a,\ 24a^4+112a^3-94a^2+18a.$
4. $9x^3+9x^2-4x-4,\ 45x^3+54x^2-20x-24.$
5. $27x^6-3x^4+6x^3-3x^2,\ 162x^6+48x^3-18x^2+6x.$
6. $20x^3-60x^2+50x-20,\ 32x^4-92x^3+68x^2-24x.$
7. $4x^2-8x-5,\ 12x^2-4x-65.$

8. $3a^3 - 5a^2x - 2ax^2,\ 9a^3 - 8a^2x - 20ax^2.$

9. $10x^3 + x^2 - 9x + 24,\ 20x^4 - 17x^2 + 48x - 3.$

10. $8x^3 - 4x^2 - 32x - 182,\ 36x^3 - 84x^2 - 111x - 126.$

11. $5x^2(12x^3 + 4x^2 + 17x - 3),\ 10x(24x^3 - 52x^2 + 14x - 1).$

12. $9x^4y - x^3y^2 - 20xy^4,\ 18x^3y - 18x^2y^2 - 2xy^3 - 8y^4.$

13. $6x^2 - x - 15,\ 9x^2 - 3x - 20.$

14. $12x^3 - 9x^2 + 5x + 2,\ 24x^2 + 10x + 1.$

15. $6x^3 + 15x^2 - 6x + 9,\ 9x^3 + 6x^2 - 51x + 36.$

16. $4x^3 - x^2y - xy^2 - 5y^3,\ 7x^3 + 4x^2y + 4xy^2 - 3y^3.$

17. $2a^3 - 2a^2 - 3a - 2,\ 3a^3 - a^2 - 2a - 16.$

18. $12y^3 + 2y^2 - 94y - 60,\ 48y^3 - 24y^2 - 348y + 30.$

19. $15x^4 + 2x^3 - 75x^2 + 5x + 2,\ 35x^4 + x^3 - 175x^2 + 30x + 1.$

20. $21x^4 - 4x^3 - 15x^2 - 2x,\ 21x^3 - 32x^2 - 54x - 7.$

144. The H. C. F. of three expressions will be obtained by finding the H. C. F. of two of them, and then of that and the third expression.

For, if A, B, and C are three expressions,

and D the highest common factor of A and B,
and E the highest common factor of D and C,

Then D contains every factor common to A and B,
and E contains every factor common to D and C.

$\therefore$ E contains every factor common to A, B, and C.

Ex. 46.

Find the H. C. F. of:

1. $2x^2 + x - 1,\ x^2 + 5x + 4,\ x^3 + 1.$

2. $y^3 - y^2 - y + 1,\ 3y^2 - 2y - 1,\ y^3 - y^2 + y - 1.$

3. $x^3-4x^2+9x-10,\ x^3+2x^2-3x+20,\ x^3+5x^2-9x+35.$

4. $x^3-7x^2+16x-12,\ 3x^3-14x^2+16x,$
$5x^3-10x^2+7x-14.$

5. $y^3-5y^2+11y-15,\ y^3-y^2+3y+5,$
$2y^3-7y^2+16y-15.$

6. $2x^2+3x-5,\ 3x^2-x-2,\ 2x^2+x-3.$

7. $x^3-1,\ x^3-x^2-x-2,\ 2x^3-x^2-x-3.$

8. $x^3-3x-2,\ 2x^3+3x^2-1,\ x^3+1.$

9. $12(x^4-y^4),\ 10(x^6-y^6),\ 8(x^4y+xy^4).$

10. $x^4+xy^3,\ x^3y+y^4,\ x^4+x^2y^2+y^4.$

Lowest Common Multiple.

145. A **common multiple** of two or more expressions is an expression which is exactly divisible by each of them.

146. The **Lowest Common Multiple** of two or more expressions is the product of all the factors of the expressions, each factor being written with its highest exponent.

147. The lowest common multiple of two expressions which have no common factor will be their product.

For brevity L. C. M. will be used for Lowest Common Multiple.

(1) Find the L. C. M. of $12a^2c$, $14bc^2$, $36ab^2$.

$$12a^2c = 2^2 \times 3a^2c,$$
$$14bc^2 = 2 \times 7bc^2,$$
$$36ab^2 = 2^2 \times 3^2ab^2.$$

$$\therefore \text{the L. C. M.} = 2^2 \times 3^2 \times 7a^2b^2c^2 = 252a^2b^2c^2.$$

(2) Find the L.C.M. of

$$2a^2+2ax,\ 6a^2-6x^2,\ 3a^2-6ax+3x^2.$$

$$2a^2+2ax \qquad = 2a(a+x),$$
$$6a^2-6x^2 \qquad = 2\times 3(a+x)(a-x),$$
$$3a^2-6ax+3x^2 = 3(a-x)^2.$$
$$\therefore \text{ the L.C.M.} = 6a(a+x)(a-x)^2.$$

Ex. 47.

Find the L.C.M. of:

1. $4a^3x,\ 6a^2x^2,\ 2ax^3.$
2. $18ax^2,\ 72ay^2,\ 12xy.$
3. $x^2,\ ax+x^2.$
4. $x^2-1,\ x^2-x.$
5. $a^2-b^2,\ a^2+ab.$
6. $2x-1,\ 4x^2-1.$
7. $a+b,\ a^3+b^3.$
8. $x^2-1,\ x^2+1,\ x^4-1.$
9. $x^2-x,\ x^3-1,\ x^3+1.$
10. $x^2-1,\ x^3-x,\ x^3-1.$
11. $2a+1,\ 4a^2-1,\ 8a^3+1.$
12. $(a+b)^2,\ a^2-b^2.$
13. $4(1+x),\ 4(1-x),\ 2(1-x^2).$
14. $x-1,\ x^2+x+1,\ x^3-1.$
15. $x^2-y^2,\ (x+y)^2,\ (x-y)^2.$
16. $x^2-y^2,\ 3(x-y)^2,\ 12(x^3+y^3).$
17. $6(x^2+xy),\ 8(xy-y^2),\ 10(x^2-y^2).$
18. $x^2+5x+6,\ x^2+6x+8.$
19. $a^2-a-20,\ a^2+a-12.$
20. $x^2+11x+30,\ x^2+12x+35.$
21. $x^2-9x-22,\ x^2-13x+22.$
22. $20(x^2-1),\ 24(x^2-x-2),\ 16(x^2+x-2).$

23. $12xy(x^2-y^2),\ 2x^2(x+y)^2,\ 3y^2(x-y)^2.$

24. $(a-b)(b-c),\ (b-c)(c-a),\ (c-a)(a-b).$

25. $(a-b)(a-c),\ (b-a)(b-c),\ (c-a)(c-b).$

26. $x^2y-xy^2,\ 3x(x-y)^2,\ 4y(x-y)^3.$

27. $(a+b)^2-(c+d)^2,\ (a+c)^2-(b+d)^2,\ (a+d)^2-(b+c)^2.$

28. $(2x-4)(3x-6),\ (x-3)(4x-8),\ (2x-6)(5x-10).$

148. When the expressions cannot be readily resolved into their factors, the expressions may be resolved by finding their H. C. F.

I. Find the L. C. M. of

$$6x^3-11x^2y+2y^3 \text{ and } 9x^3-22xy^2-8y^3.$$

$6x^3-11x^2y\qquad+2y^3$	$9x^3-22xy^2-\ \ 8y^3$	3
$6x^3-\ \ 8x^2y-4xy^2$	2	
$-\ \ 3x^2y+4xy^2+2y^3$	$18x^3-44xy^2-16y^3$	
$-\ \ 3x^2y+4xy^2+2y^3$	$18x^3-33x^2y+\ \ 6y^3$	
	$11y)\ 33x^2y-44xy^2-22y^3$	
	$3x^2\ -\ 4xy\ -\ 2y^2$	$2x-y$

Hence, $6x^3-11x^2y+2y^3=(2x-y)\ (3x^2-4xy-2y^2),$

and $\qquad 9x^3-22xy^2-8y^3=(3x+4y)(3x^2-4xy-2y^2).$

$\therefore$ the L. C. M. $=(2x-y)(3x+4y)(3x^2-4xy-2y^2).$

In this example we find the H. C. F. of the given expressions, and divide each of them by the H. C. F.

149. To find the L. C. M. of *three* expressions, A, B, C. Find M, the L. C. M. of A and B; then the L. C. M. of M and C is the L. C. M. required.

Ex. 48.

Find the L. C. M. of:

1. $6x^2 - x - 2,\ 21x^2 - 17x + 2,\ 14x^2 + 5x - 1.$
2. $x^2 - 1,\ x^2 + 2x - 3,\ 6x^2 - x - 2.$
3. $x^3 - 27,\ x^2 - 15x + 36,\ x^3 - 3x^2 - 2x + 6.$
4. $5x^2 + 19x - 4,\ 10x^2 + 13x - 3.$
5. $12x^2 + xy - 6y^2,\ 18x^2 + 18xy - 20y^2.$
6. $x^4 - 2x^3 + x,\ 2x^4 - 2x^3 - 2x - 2.$
7. $12x^2 + 2x - 4,\ 12x^2 - 42x - 24,\ 12x^2 - 28x - 24.$
8. $x^3 - 6x^2 + 11x - 6,\ x^3 - 9x^2 + 26x - 24,$ $x^3 - 8x^2 + 19x - 12.$
9. $x^3 + 2x^2y - xy^2 - 2y^3,\ x^3 - 2x^2y - xy^2 + 2y^3.$
10. $1 + p + p^2,\ 1 - p + p^2,\ 1 + p^2 + p^4.$
11. $(1 - a),\ (1 - a)^2,\ (1 - a)^3.$
12. $(a + c)^2 - b^2,\ (a + b)^2 - c^2,\ (b + c)^2 - a^2.$
13. $b^4 - 2b^3 + b^2 - 8b + 8,\ 4b^3 - 12b^2 + 9b - 1.$

CHAPTER VIII.

FRACTIONS.

150. The expression $\frac{a}{b}$ is employed to indicate that a units are divided into b equal parts, and that *one* of these parts is taken;

or, that *one* unit is divided into b equal parts, and that a of these parts are taken.

151. The expression $\frac{a}{b}$ is called a **fraction.** a is the **numerator,** and b the **denominator.**

152. The numerator and denominator are called the **terms** of the fraction.

153. The denominator shows into how many equal parts the unit is divided, and therefore *names* the part; and the numerator shows how many of these parts are taken.

It will be observed that a letter written *above* the line in a fraction serves a very different purpose from that of a letter written *below* the line.

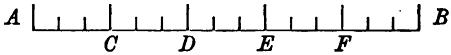
A letter written above the line denotes **number;**

A letter written below the line denotes **name.**

154. Every whole number may be written in the form of a fraction with unity for its denominator; thus, $a = \frac{a}{1}$.

To Reduce a Fraction to its Lowest Terms.

155. Let the line AB be divided into 5 equal parts, at the points C, D, E, F.



Then AF is $\frac{4}{5}$ of AB. (1)

Now let each of the parts be subdivided into 3 equal parts.

Then AB contains 15 of these subdivisions, and AF contains 12 of these subdivisions.

$\therefore AF$ is $\frac{12}{15}$ of AB. (2)

Comparing (1) and (2), it is evident that $\frac{4}{5} = \frac{12}{15}$.

In general:

If we suppose AB to be divided into b equal parts, and that AF contains a of these parts,

Then AF is $\frac{a}{b}$ of AB. (3)

Now, if we suppose each of the parts to be subdivided into c equal parts,

Then AB contains bc of these subdivisions, and AF contains ac of these subdivisions.

$\therefore AF$ is $\frac{ac}{bc}$ of AB. (4)

Comparing (3) and (4), it is evident that

$$\frac{a}{b} = \frac{ac}{bc}.$$

Since $\frac{ac}{bc}$ is obtained by multiplying by c both terms of the fraction $\frac{a}{b}$,

and, conversely, $\frac{a}{b}$ is obtained by dividing by c both terms of the fraction $\frac{ac}{bc}$, it follows that

I. If the numerator and denominator of a fraction be multiplied by the same number, the value of the fraction is not altered.

II. If the numerator and denominator be divided by the same number, the value of the fraction is not altered.

Hence, to reduce a fraction to lower terms,

Divide the numerator and denominator by any common factor.

156. A fraction is expressed in its lowest terms when both numerator and denominator are divided by their H. C. F.

Reduce the following fractions to their lowest terms:

(1) $\frac{a^3-x^3}{a^2-x^2}=\frac{(a-x)(a^2+ax+x^2)}{(a-x)(a+x)}=\frac{a^2+ax+x^2}{a+x}.$

(2) $\frac{a^2+7a+10}{a^2+5a+6}=\frac{(a+5)(a+2)}{(a+3)(a+2)}=\frac{a+5}{a+3}.$

(3) $\frac{6x^2-5x-6}{8x^2-2x-15}=\frac{(2x-3)(3x+2)}{(2x-3)(4x+5)}=\frac{3x+2}{4x+5}.$

(4) $\frac{a^3-7a^2+16a-12}{3a^3-14a^2+16a}.$

Since in Ex. (4) no common factor can be determined by inspection, it is necessary to find the H. C. F. of the numerator and denominator by the method of division.

Suppress the factor a of the denominator and proceed to divide:

$$
\begin{array}{r|r|c}
a^3 - 7a^2 + 16a - 12 & 3a^2 - 14a + 16 & a - 7 \\
3 & \underline{3a^2 - 6a} & \\
3a^3 - 21a^2 + 48a - 36 & -8a + 16 & 3a - 8 \\
\underline{3a^3 - 14a^2 + 16a} & \underline{-8a + 16} & \\
-7a^2 + 32a - 36 & & \\
\underline{3} & & \\
-21a^2 + 96a - 108 & & \\
\underline{-21a^2 + 98a - 112} & & \\
-2\,\underline{)\,-2a + 4} & & \\
a - 2 & &
\end{array}
$$

$$\therefore \text{ the H. C. F.} = a - 2.$$

Now, if $a^3 - 7a^2 + 16a - 12$ be divided by $a - 2$, the result is $a^2 - 5a + 6$; and if $3a^3 - 14a^2 + 16a$ be divided by $a - 2$, the result is $3a^2 - 8a$.

$$\therefore \frac{a^3 - 7a^2 + 16a - 12}{3a^3 - 14a^2 + 16a} = \frac{a^2 - 5a + 6}{3a^2 - 8a}.$$

157. When common factors cannot be determined by inspection, the H. C. F. must be found by the method of division.

Ex. 49.

Reduce to lowest terms:

1. $\dfrac{x^2 - 1}{4x(x + 1)}$.

2. $\dfrac{x^2 - 9x + 20}{x^2 - 7x + 12}$.

3. $\dfrac{x^2 - 2x - 3}{x^2 - 10x + 21}$.

4. $\dfrac{x^4 + x^2 + 1}{x^2 + x + 1}$.

5. $\dfrac{x^6 + 2x^3y^3 + y^6}{x^6 - y^6}$.

6. $\dfrac{a^3 + 1}{a^3 + 2a^2 + 2a + 1}$.

7. $\dfrac{a^2 - a - 20}{a^2 + a - 12}$.

8. $\dfrac{x^3 - 4x^2 + 9x - 10}{x^3 + 2x^2 - 3x + 20}$.

9. $\dfrac{x^3 - 5x^2 + 11x - 15}{x^3 - x^2 + 3x + 5}$.

10. $\dfrac{x^4 + x^3y + xy^3 - y^4}{x^4 - x^3y - xy^3 - y^4}$.

11. $\dfrac{a^3+4a^2-5}{a^3-3a+2}$.

12. $\dfrac{3x^2+2x-1}{x^3+x^2-x-1}$.

13. $\dfrac{x^3-3x^2+4x-2}{x^3-x^2-2x+2}$.

14. $\dfrac{4x^2-12ax+9a^2}{8x^3-27a^3}$.

15. $\dfrac{15a^2+ab-2b^2}{9a^2+3ab-2b^2}$.

16. $\dfrac{a^2-b^2-2bc-c^2}{a^2+2ab+b^2-c^2}$.

To Reduce a Fraction to an Integral or Mixed Expression.

Change $\dfrac{x^3+1}{x-1}$ to a mixed expression.

$(x^3+1)\div(x-1)=x^2+x+1+\dfrac{2}{x-1}$. Hence,

158. *If the degree of the numerator of a fraction equals or exceeds that of the denominator, the fraction may be changed to the form of a mixed or integral expression by dividing the numerator by the denominator.*

The quotient will be the integral expression, the remainder (if any) will be the numerator, and the divisor the denominator, of the fractional expression.

Ex. 50.

Change to integral or mixed expressions:

1. $\dfrac{x^2-2x+1}{x-1}$.

2. $\dfrac{3x^2+2x+1}{x+4}$.

3. $\dfrac{3x^2+6x+5}{x+4}$.

4. $\dfrac{a^2-ax+x^2}{a+x}$.

5. $\dfrac{2x^2+5}{x-3}$.

6. $\dfrac{10a^2-17ax+10x^2}{5a-x}$.

7. $\dfrac{16(3x^2+1)}{4x-1}$.

8. $\dfrac{2x^2-5x-2}{x-4}$.

9. $\dfrac{a^2+b^2}{a-b}$.

10. $\dfrac{5x^3-x^2+5}{5x^2+4x-1}$.

TO REDUCE A MIXED EXPRESSION TO THE FORM OF A FRACTION.

159. In arithmetic $5\frac{3}{4}$ means $5+\frac{3}{4}$.

But in algebra the fraction connected with the integral expression, as well as the integral expression, may be positive or negative; so that a mixed expression may occur in any one of the following forms:

$$n+\frac{a}{b};\quad n-\frac{a}{b};\quad -n+\frac{a}{b};\quad -n-\frac{a}{b}.$$

Change $n+\frac{a}{b}$ to a fractional form.

Since there are b *bths* in 1, in n there will be n times b *bths*, that is, nb *bths*, which, with the additional a *bths*, make $nb+a$ *bths*.

$$\therefore n+\frac{a}{b}=\frac{nb+a}{b}.$$

In like manner:

$$n-\frac{a}{b}=\frac{nb-a}{b};$$

$$-n+\frac{a}{b}=\frac{-nb+a}{b};$$

and $$-n-\frac{a}{b}=\frac{-nb-a}{b}.$$ Hence,

160. To reduce a mixed expression to a fractional form,

Multiply the integral expression by the denominator, to the product annex the numerator, and under the result write the denominator.

161. It will be seen that the *sign* before the fraction is transferred to the *numerator* when the mixed expression is reduced to the fractional form, for the denominator shows only what *part* of the numerator is to be added or subtracted.

The dividing line has the force of a vinculum or parenthesis affecting the numerator; therefore if a *minus sign* precede the dividing line, and this line be removed, the *sign of every term of the numerator must be changed.* Thus,

$$n - \frac{a-b}{c} = \frac{cn-(a-b)}{c}$$

$$= \frac{cn-a+b}{c}.$$

(1) Change to fractional form $x - 1 + \frac{x-1}{x}$.

$$x - 1 + \frac{x-1}{x}$$

$$= \frac{x^2 - x + (x-1)}{x}$$

$$= \frac{x^2 - x + x - 1}{x}$$

$$= \frac{x^2 - 1}{x}.$$

(2) Change to fractional form $x - 1 - \frac{x-1}{x}$.

$$x-1-\frac{x-1}{x}$$
$$=\frac{x^2-x-(x-1)}{x}$$
$$=\frac{x^2-x-x+1}{x}$$
$$=\frac{x^2-2x+1}{x}.$$

Ex. 51.

Change to fractional form:

1. $1-\frac{x-y}{x+y}$.
2. $1+\frac{x-y}{x+y}$.
3. $3x-\frac{1+2x^2}{x}$.
4. $a-x+\frac{a^2+x^2}{a-x}$.
5. $5a-2b-\frac{3a^2-4b^2}{5a-6b}$.
6. $a+b-\frac{a^2+b^2}{a+b}$.
7. $7a-\frac{2-3a+4a^2}{5-6a}$.
8. $3x-\frac{5ax-3}{2a}$.
9. $\frac{a+b}{a-b}+1$.
10. $\frac{a-b}{a+b}-1$.
11. $\frac{2x^2}{x+y}-(x+y)$.
12. $\frac{5a-12x}{4}+6a+3x$.
13. $a-1+\frac{1}{a+1}$.
14. $x+5-\frac{2x-15}{x-3}$.
15. $2a-b-\frac{2ab}{a+b}$.
16. $3x-10+\frac{41}{x+4}$.
17. $x^2+x+1+\frac{2}{x-1}$.
18. $x^3-3x-\frac{3x(3-x)}{x-2}$.
19. $a^2-2ax+4x^2-\frac{6x^3}{a+2x}$.
20. $x-a+y+\frac{a^2-ay+y^2}{x+a}$.

Lowest Common Denominator.

162. To reduce fractions to equivalent fractions having the lowest common denominator:

Reduce $\frac{3x}{4a^2}$, $\frac{2y}{3a}$, and $\frac{5}{6a^3}$ to equivalent fractions having the lowest common denominator.

The L. C. M. of $4a^2$, $3a$, and $6a^3 = 12a^3$.

If both terms of $\frac{3x}{4a^2}$ be multiplied by $3a$, the value of the fraction will not be altered, but the form will be changed to $\frac{9ax}{12a^3}$; if both terms of $\frac{2y}{3a}$ be multiplied by $4a^2$, the equivalent fraction $\frac{8a^2y}{12a^3}$ is obtained; and, if both terms of $\frac{5}{6a^3}$ be multiplied by 2, the equivalent fraction $\frac{10}{12a^3}$ is obtained.

Hence, $\frac{3x}{4a^2}, \frac{2y}{3a}, \frac{5}{6a^3}$,

are equal to $\frac{9ax}{12a^3}, \frac{8a^2y}{12a^3}, \frac{10}{12a^3}$, respectively.

The multipliers $3a$, $4a^2$, and 2 are obtained by dividing $12a^3$, the L. C. M. of the denominators, by the respective denominators of the given fractions.

163. Therefore, to reduce fractions to equivalent fractions having the lowest common denominator,

Find the L. C. M. *of the denominators.*

Divide the L. C. M. *by the denominator of each fraction.*

Multiply the first numerator by the first quotient, the second by the second quotient, and so on.

The products will be the numerators of the equivalent fractions.

The L. C. M. *of the given denominators will be the denominator of each of the equivalent fractions.*

Ex. 52.

Reduce to equivalent fractions with the lowest common denominator:

1. $\dfrac{3x-7}{6}, \dfrac{4x-9}{18}.$

2. $\dfrac{2x-4y}{5x^2}, \dfrac{3x-8y}{10x}.$

3. $\dfrac{4a-5c}{5ac}, \dfrac{3a-2c}{12a^2c}.$

4. $\dfrac{5}{1-x}, \dfrac{6}{1-x^2}.$

5. $\dfrac{1}{(a-b)(b-c)}, \dfrac{1}{(a-b)(a-c)}.$

6. $\dfrac{4x^2}{3(a+b)}, \dfrac{xy}{6(a^2-b^2)}.$

7. $\dfrac{8x+2}{x-2}, \dfrac{2x-1}{3x-6}, \dfrac{3x+2}{5x-10}.$

8. $\dfrac{a-bm}{mx}, 1, \dfrac{c-bn}{nx}.$

Addition and Subtraction of Fractions.

164. To add fractions:

Reduce the fractions to equivalent fractions having the lowest common denominator.

Add the numerators of the equivalent fractions.

Write the result over the lowest common denominator.

165. To subtract one fraction from another:

Reduce the fractions to equivalent fractions having the lowest common denominator.

Subtract the numerator of the subtrahend from the numerator of the minuend.

Write the result over the lowest common denominator.

(1) Simplify $\dfrac{4x+7}{5}+\dfrac{3x-4}{15}.$

The lowest common denominator (L. C. D.) = 15.

The multipliers are 3 and 1 respectively.

$$\begin{aligned} 12x + 21 &= \text{1st numerator,} \\ 3x - 4 &= \text{2d numerator.} \\ \hline 15x + 17 &= \text{sum of numerators.} \end{aligned}$$

$$\therefore \frac{4x+7}{5} + \frac{3x-4}{15} = \frac{15x+17}{15}.$$

(2) Simplify $\dfrac{3a-4b}{7} - \dfrac{2a-b+c}{3} + \dfrac{13a-4c}{12}.$

The L. C. D. = 84.

The multipliers are 12, 28, and 7 respectively.

$$\begin{aligned} 36a - 48b \phantom{{}-28c} &= \text{1st numerator,} \\ -56a + 28b - 28c &= \text{2d numerator,} \\ 91a \phantom{{}+28b} - 28c &= \text{3d numerator.} \\ \hline 71a - 20b - 56c &= \text{sum of numerators.} \end{aligned}$$

$$\therefore \frac{3a-4b}{7} - \frac{2a-b+c}{3} + \frac{13a-4c}{12} = \frac{71a-20b-56c}{84}.$$

Since the *minus sign* precedes the second fraction, the signs of all the terms of the numerator of this fraction are changed after being multiplied by 28.

Ex. 53.

Simplify:

1. $\dfrac{3x-2y}{5x} + \dfrac{5x-7y}{10x} + \dfrac{8x+2y}{25}.$

2. $\dfrac{4x^2-7y^2}{3x^2} + \dfrac{3x-8y}{6x} + \dfrac{5-2y}{12}.$

3. $\dfrac{4a^2+5b^2}{2b^2} + \dfrac{3a+2b}{5b} + \dfrac{7-2a}{9}.$

4. $\frac{4x+5}{3}-\frac{3x-7}{5x}+\frac{9}{12x^2}$.

5. $\frac{4x-3y}{7}+\frac{3x+7y}{14}-\frac{5x-2y}{21}+\frac{9x+2y}{42}$.

6. $\frac{3xy-4}{x^2y^2}-\frac{5y^2+7}{xy^3}-\frac{6x^2-11}{x^3y}$.

7. $\frac{a^2-2ac+c^2}{a^2c^2}-\frac{b^2-2bc+c^2}{b^2c^2}$.

8. $\frac{5a^3-2}{8a^3}-\frac{3a^2-a}{8}$.

9. $\frac{a-b}{c}+\frac{b-c}{a}+\frac{c-a}{b}+\frac{ab^2+bc^2+ca^2}{abc}$.

10. $\frac{1}{2x^2y}-\frac{1}{6y^2z}-\frac{1}{2xz^2}+\frac{2x-z}{4x^2z^2}+\frac{y-2z}{4x^2yz}$.

Simplify $\frac{x-y}{x+y}+\frac{x+y}{x-y}$.

The L. C. D. $=x^2-y^2$.

The multipliers are $x-y$ and $x+y$ respectively.

$$
\begin{array}{ll}
x^2-2xy+y^2 & = \text{1st numerator,} \\
x^2+2xy+y^2 & = \text{2d numerator.} \\
\hline
2x^2 \qquad\quad +2y^2 & = \text{sum of numerators.} \\
\text{or,} \quad 2(x^2+y^2) & = \text{“ \quad “ \quad “}
\end{array}
$$

$$\therefore \frac{x-y}{x+y}+\frac{x+y}{x-y}=\frac{2(x^2+y^2)}{x^2-y^2}.$$

Ex. 54.

Simplify:

1. $\dfrac{1}{x-6}+\dfrac{1}{x+5}$.

2. $\dfrac{1}{x-7}-\dfrac{1}{x-3}$.

3. $\dfrac{1}{1+x}+\dfrac{1}{1-x}$.

4. $\dfrac{1}{1-x}-\dfrac{2}{1-x^2}$.

5. $\dfrac{1}{x-y}+\dfrac{x}{(x-y)^2}$.

6. $\dfrac{1}{2a(a+x)}+\dfrac{1}{2a(a-x)}$.

7. $\dfrac{a}{(a+b)b}-\dfrac{b}{(a-b)a}$.

8. $\dfrac{5}{2x(x-1)}-\dfrac{3}{4x(x-2)}$.

9. $\dfrac{1+x}{1+x+x^2}-\dfrac{1-x}{1-x+x^2}$.

10. $\dfrac{2ax-3by}{2xy(x-y)}-\dfrac{2ax+3by}{2xy(x+y)}$.

(1) Simplify $\dfrac{2a+b}{a-b}-\dfrac{2a-b}{a+b}-\dfrac{6ab}{a^2-b^2}$.

The L. C. D. $=(a-b)(a+b)$.

The multipliers are $a+b$, $a-b$, and 1, respectively.

$$\begin{array}{rl} 2a^2+3ab+b^2 & = \text{1st numerator,} \\ -2a^2+3ab-b^2 & = \text{2d numerator,} \\ -6ab & = \text{3d numerator.} \\ \hline 0 & = \text{sum of numerators.} \end{array}$$

$$\therefore \frac{2a+b}{a-b}-\frac{2a-b}{a+b}-\frac{6ab}{a^2-b^2}=0.$$

(2) Simplify $\dfrac{y^2}{x^2-y^2}-\dfrac{x-y}{x+y}+1+\dfrac{2xy}{x^2+y^2}$.

The L. C. D. $=(x+y)(x-y)(x^2+y^2)$.

The multipliers are x^2+y^2, $(x-y)(x^2+y^2)$, $(x+y)(x-y)(x^2+y^2)$, $(x+y)(x-y)$, respectively.

$$\begin{array}{rcl}
x^2y^2 \qquad\qquad + y^4 &=& \text{1st numerator,} \\
-x^4+2x^3y-2x^2y^2+2xy^3-y^4 &=& \text{2d numerator,} \\
x^4 \qquad\qquad\qquad\qquad -y^4 &=& \text{3d numerator,} \\
2x^3y \qquad -2xy^3 \qquad &=& \text{4th numerator.} \\
\hline
4x^3y-\ x^2y^2 \qquad\quad -y^4 &=& \text{sum of numerators.}
\end{array}$$

$$\therefore \text{Sum of fractions} = \frac{4x^3y-x^2y^2-y^4}{x^4-y^4}.$$

Ex. 55.

Simplify:

1. $\dfrac{1}{1+a}+\dfrac{1}{1-a}+\dfrac{2a}{1-a^2}$.

2. $\dfrac{1}{1-x}-\dfrac{1}{1+x}+\dfrac{2x}{1+x^2}$.

3. $\dfrac{x}{1-x}-\dfrac{x^2}{1-x}+\dfrac{x}{1+x^2}$.

4. $\dfrac{x}{y}+\dfrac{y}{x+y}+\dfrac{x^2}{x^2+xy}$.

5. $\dfrac{3}{x-a}+\dfrac{4a}{(x-a)^2}-\dfrac{5a^2}{(x-a)^3}$.

6. $\dfrac{1}{x-1}-\dfrac{1}{x+2}-\dfrac{3}{(x+1)(x+2)}$.

7. $\dfrac{a-b}{(b+c)(c+a)}+\dfrac{b-c}{(c+a)(a+b)}+\dfrac{c-a}{(a+b)(b+c)}$.

8. $\dfrac{x-a}{x-b}+\dfrac{x-b}{x-a}-\dfrac{(a-b)^2}{(x-a)(x-b)}$.

9. $\dfrac{x+y}{y}-\dfrac{2x}{x+y}+\dfrac{x^2y-x^3}{y(x^2-y^2)}$.

10. $\dfrac{a+b}{(b-c)(c-a)}+\dfrac{b+c}{(c-a)(a-b)}+\dfrac{c+a}{(a-b)(b-c)}$.

11. $\dfrac{a}{a-x}-\dfrac{x}{a+2x}-\dfrac{a^2+x^2}{(a-x)(a+2x)}.$

12. $\dfrac{3}{(a-b)(b-c)}-\dfrac{4}{(a-b)(a-c)}+\dfrac{6}{(a-c)(b-c)}.$

13. $\dfrac{x-2y}{x(x-y)}-\dfrac{2x+y}{y(x+y)}-\dfrac{2x}{x^2-y^2}.$

14. $\dfrac{3x}{(x+y)^2}-\dfrac{x+2y}{x^2-y^2}+\dfrac{3y}{(x-y)^2}.$

15. $\dfrac{a-c}{(a+b)^2-c^2}-\dfrac{a-b}{(a+c)^2-b^2}.$

166. Since $$\frac{ab}{b}=a, \text{ and } \frac{-ab}{-b}=a,$$

it is evident that if the signs of both numerator and denominator be changed, the value of the fraction is not altered.

Again, $$\frac{a-b}{c-d}=\frac{-(a-b)}{-(c-d)}=\frac{-a+b}{-c+d}=\frac{b-a}{d-c}.$$

Therefore, if the numerator or denominator be a compound expression, or if both be compound expressions, the sign of every term in the denominator may be changed, provided the sign of every term in the numerator be also changed.

Since the change of the sign before the fraction is equivalent to the change of the sign before every term of the numerator of the fraction, *the sign before every term of the denominator may be changed, provided the sign before the fraction be changed.*

Since, also, the product of $+a$ multiplied by $+b$ is ab, and the product of $-a$ multiplied by $-b$ is ab, the signs of *two factors*, or of *any even number of factors*, of the de-

nominator of a fraction may be changed without altering the value of the fraction.

By the application of these principles, fractions may often be changed to a more simple form for addition or subtraction.

(1) Simplify $\quad \dfrac{2}{x}-\dfrac{3}{2x-1}+\dfrac{2x-3}{1-4x^2}.$

Change the signs before the terms of the denominator of the third fraction, and change the sign before the fraction.

The result is,

$$\frac{2}{x}-\frac{3}{2x-1}-\frac{2x-3}{4x^2-1},$$

in which the several denominators are written in symmetrical form.

The L. C. D. $=x(2x-1)(2x+1)$.

$$\begin{aligned} 8x^2-2\quad &= \text{1st numerator,}\\ -6x^2-3x &= \text{2d numerator,}\\ \underline{-2x^2+3x} &= \text{3d numerator.}\\ -2\quad &= \text{sum of numerators.} \end{aligned}$$

$$\therefore \text{Sum of the fractions} = \frac{-2}{x(2x-1)(2x+1)}.$$

(2) Simplify

$$\frac{1}{a(a-b)(a-c)}+\frac{1}{b(b-a)(b-c)}+\frac{1}{c(c-a)(c-b)}.$$

Change the sign of the factor $(b-a)$ in the denominator of the second fraction, and change the sign before the fraction.

Then change the signs of the factors $(c-a)$ and $(c-b)$ in the denominator of the third fraction.

The result is,

$$\frac{1}{a(a-b)(a-c)}-\frac{1}{b(a-b)(b-c)}+\frac{1}{c(a-c)(b-c)},$$

in which the factors of the several denominators are written in symmetrical form.

The L. C. D. $= abc(a-b)(a-c)(b-c)$.

$$
\begin{array}{rl}
bc(b-c) = b^2c - bc^2 & = \text{1st numerator,} \\
-ac(a-c) = -a^2c + ac^2 & = \text{2d numerator,} \\
ab(a-b) = a^2b - ab^2 & = \text{3d numerator.} \\
\hline
a^2b - a^2c - ab^2 + ac^2 + b^2c - bc^2 & = \text{sum of numerators,}
\end{array}
$$

$$
\begin{aligned}
&= a^2(b-c) - a(b^2-c^2) + bc(b-c) \\
&= [a^2 - a(b+c) + bc][b-c] \\
&= [a^2 - ab - ac + bc][b-c] \\
&= [(a^2-ac) - (ab-bc)][b-c] \\
&= [a(a-c) - b(a-c)][b-c] \\
&= (a-b)(a-c)(b-c).
\end{aligned}
$$

$$
\therefore \text{Sum of the fractions} = \frac{(a-b)(a-c)(b-c)}{abc(a-b)(a-c)(b-c)} = \frac{1}{abc}.
$$

Ex. 56.

Simplify:

1. $\dfrac{x}{x-y} + \dfrac{x-y}{y-x}$.

2. $\dfrac{3+2x}{2-x} + \dfrac{3x-2}{2+x} + \dfrac{16x-x^2}{x^2-4}$.

3. $\dfrac{x^2}{x^2-1} + \dfrac{x}{x+1} - \dfrac{x}{1-x}$.

4. $\dfrac{4}{3-3y^2} + \dfrac{1}{2-2y} + \dfrac{1}{6y+6}$.

5. $\dfrac{1}{(2-m)(3-m)} - \dfrac{2}{(m-1)(m-3)} + \dfrac{1}{(m-1)(m-2)}$.

6. $\dfrac{1}{(b-a)(x+a)}+\dfrac{1}{(a-b)(x+b)}.$

7. $\dfrac{a^2+b^2}{a^2-b^2}+\dfrac{2ab^2}{b^3-a^3}+\dfrac{2a^2b}{a^3+b^3}.$

8. $\dfrac{b-a}{x-b}-\dfrac{a-2b}{b+x}-\dfrac{3x(a-b)}{b^2-x^2}.$

9. $\dfrac{3+2x}{2-x}-\dfrac{2-3x}{2+x}+\dfrac{16x-x^2}{x^2-4}.$

10. $\dfrac{3}{1-2x}-\dfrac{7}{1+2x}-\dfrac{4-20x}{4x^2-1}.$

11. $\dfrac{a+b}{(b-c)(c-a)}+\dfrac{b+c}{(b-a)(a-c)}+\dfrac{c+a}{(a-b)(b-c)}.$

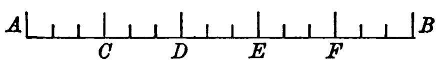
12. $\dfrac{a^2-bc}{(a-b)(a-c)}+\dfrac{b^2+ac}{(b+c)(b-a)}+\dfrac{c^2+ab}{(c-a)(c+b)}.$

Multiplication of Fractions.

167. Hitherto in fractions, equal parts of one or more *units* have been taken. But it is often necessary to take equal parts of *fractions of units*.

Suppose it is required to take $\frac{2}{3}$ of $\frac{4}{5}$ of a unit.

Let the line AB represent the unit of length.

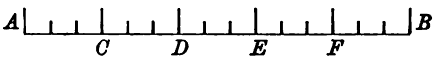


Suppose AB divided into 5 equal parts, at C, D, E, and F, and each of these parts to be subdivided into 3 equal subdivisions.

Then one of the parts, as AC, will contain 3 of these subdivisions, and the whole line AB will contain 15 of these subdivisions.

That is, $\frac{1}{3}$ of $\frac{1}{5}$ of the line will be $\frac{1}{15}$ of the line; $\frac{1}{3}$ of $\frac{4}{5}$ will be $\frac{1}{15}+\frac{1}{15}+\frac{1}{15}+\frac{1}{15}$, or $\frac{4}{15}$, of the line; and $\frac{2}{3}$ of $\frac{4}{5}$ will be *twice* $\frac{4}{15}$, or $\frac{8}{15}$, of the line.

Suppose it is required to take $\frac{c}{d}$ of $\frac{a}{b}$ of the line AB.



Let the line AB be divided into b equal parts, and let each of these parts be subdivided into d equal subdivisions.

Then the whole line will contain bd of these subdivisions, and one of these subdivisions will be $\frac{1}{bd}$ of the line.

If one of the subdivisions be taken from each of a parts, they will together be $\frac{a}{bd}$ of the line. That is,

$$\frac{1}{d} \text{ of } \frac{a}{b} = \frac{1}{bd} + \frac{1}{bd} + \frac{1}{bd} \cdots\cdots \text{ taken } a \text{ times, } = \frac{a}{bd},$$

and $\frac{c}{d}$ of $\frac{a}{b}$ will be c *times* $\frac{a}{bd}$, or $\frac{ac}{bd}$ of the line.

Therefore, to find a fraction of a fraction,

Find the product of the numerators for the numerator of the product, and of the denominators for the denominator of the product.

168. Now, $\frac{c}{d} \times \frac{a}{b}$ means $\frac{c}{d}$ of $\frac{a}{b}$.

Therefore, to find the product of two fractions,

Find the product of the numerators for the numerator of the product, and of the denominators for the denominator of the product.

The same rule will hold when more than two fractions are taken.

If a factor exist in both a numerator and a denominator, it may be cancelled; for the cancelling of a common factor *before* the multiplication is evidently equivalent to cancelling it *after* the multiplication; and this may be done by § 155.

Division of Fractions.

169. Multiplying by the reciprocal of a number is equivalent to dividing by the number. Thus, multiplying by $\frac{1}{4}$ is equivalent to dividing by 4.

The reciprocal of a fraction is the fraction with its terms interchanged.

Thus, the reciprocal of $\frac{2}{3}$ is $\frac{3}{2}$, for $\frac{3}{2} \times \frac{2}{3} = 1$. § 42.

Therefore, to divide by a fraction,

Interchange the terms of the fraction and multiply by the resulting fraction. Thus:

(1) $\dfrac{2a}{3x^3} \div \dfrac{1}{3x} = \dfrac{2a}{3x^3} \times \dfrac{3x}{1} = \dfrac{2a}{x}.$

The common factor cancelled is $3x$.

(2) $\dfrac{14x^2}{27y^2} \div \dfrac{7x}{9y} = \dfrac{14x^2}{27y^2} \times \dfrac{9y}{7x} = \dfrac{2x}{3y}.$

The common factors cancelled are $9y$ and $7x$.

(3) $$\frac{ax}{(a-x)^2} \div \frac{ab}{a^2-x^2} = \frac{ax}{(a-x)(a-x)} \times \frac{(a+x)(a-x)}{ab}$$
$$= \frac{x(a+x)}{b(a-x)}.$$

The common factors cancelled are a and $a - x$.

If the divisor be an integral expression, it may be changed to the fractional form. § 154.

Ex. 57.

1. $\dfrac{a}{bx} \times \dfrac{cx}{d}.$

2. $\dfrac{2x}{a} \times \dfrac{3ab}{c} \times \dfrac{3ac}{2b}.$

3. $\dfrac{3p}{2p-2} \div \dfrac{2p}{p-1}.$

4. $\dfrac{8x^4y}{15ab^3} \div \dfrac{2x^3}{3ab^2}.$

5. $\dfrac{8a^2b^3}{45x^2y} \times \dfrac{15xy^3}{24a^3b^2}.$

6. $\dfrac{9x^2y^2z}{10a^2b^2c} \times -\dfrac{20a^3b^3c}{18xy^2z}.$

7. $\dfrac{3x^3y}{4xz^2} \times \dfrac{5y^2z}{6xy} \times -\dfrac{12x^3}{2xy^2}.$

8. $\dfrac{9m^2n^2}{8p^3q^3} \times \dfrac{5p^2q}{2xy} \times \dfrac{24x^2y^2}{90mn}.$

9. $\dfrac{25k^3m^2}{14n^2q^2} \times \dfrac{70n^3q}{75p^2m} \times \dfrac{3pm}{4k^2n}.$

10. $\dfrac{a-b}{a^2+ab} \times \dfrac{a^2-b^2}{a^2-ab}.$

11. $\dfrac{a^2+b^2}{a^2-b^2} \div \dfrac{a-b}{a+b}.$

12. $\dfrac{x^2+x-2}{x^2-7x} \times \dfrac{x^2-13x+42}{x^2+2x}.$

13. $\dfrac{x^2-11x+30}{x^2-6x+9} \times \dfrac{x^2-3x}{x^2-5x}.$

14. $\dfrac{a^3-x^3}{a^3+x^3} \times \dfrac{(a+x)^2}{(a-x)^2}.$

15. $\dfrac{2a(x^2-y^2)^2}{cx} \times \dfrac{x^3}{(x-y)(x+y)^2}.$

16. $\dfrac{a^2+2ab}{a^2+4b^2} \times \dfrac{ab-2b^2}{a^2-4b^2}.$

17. $\dfrac{x^2-4}{x^2+5x} \times \dfrac{x^2-25}{x^2+2x}.$

18. $\dfrac{x^2+xy}{x-y} \times \dfrac{(x-y)^2}{x^4-y^4}.$

19. $\dfrac{m^2-n^2}{c^3+d^3} \div \dfrac{n-m}{c+d}.$

20. $\dfrac{a^2-4a+3}{a^2-5a+4} \times \dfrac{a^2-9a+20}{a^2-10a+21} \times \dfrac{a^2-7a}{a^2-5a}.$

21. $\dfrac{b^2-7b+6}{b^2+3b-4}\times\dfrac{b^2+10b+24}{b^2-14b+48}\div\dfrac{b^2+6b}{b^3-8b^2}.$

22. $\dfrac{x^2-y^2}{x^2-3xy+2y^2}\times\dfrac{xy-2y^2}{x^2+xy}\times\dfrac{x^2-xy}{(x-y)^2}.$

23. $\dfrac{a^3-3a^2b+3ab^2-b^3}{a^2-b^2}\div\dfrac{2ab-2b^2}{3}\times\dfrac{a^2+ab}{a-b}.$

24. $\dfrac{(a+b)^2-c^2}{a^2-(b-c)^2}\div\dfrac{c^2-(a+b)^2}{c^2-(a-b)^2}.$

25. $\dfrac{(x-a)^2-b^2}{(x-b)^2-a^2}\times\dfrac{x^2-(b-a)^2}{x^2-(a-b)^2}.$

26. $\dfrac{(a+b)^2-(c+d)^2}{(a+c)^2-(b+d)^2}\div\dfrac{(a-c)^2-(d-b)^2}{(a-b)^2-(d-c)^2}.$

27. $\dfrac{x^2-2xy+y^2-z^2}{x^2+2xy+y^2-z^2}\times\dfrac{x+y-z}{x-y+z}.$

Complex Fractions.

170. A **complex fraction** is one which has a fraction in the numerator or in the denominator, or in both.

171. A fraction may be regarded as the *quotient* of the numerator divided by the denominator.

This is the simplest meaning of a complex fraction.

Therefore, to simplify a complex fraction,

Divide the numerator by the denominator.

(1) Simplify $\dfrac{\frac{1}{3}}{\frac{1}{2}}.$

$$\frac{\frac{1}{3}}{\frac{1}{2}}=\tfrac{1}{3}\div\tfrac{1}{2}=\tfrac{1}{3}\times\tfrac{2}{1}=\tfrac{2}{3}.$$

(2) Simplify $\dfrac{2\frac{2}{3}}{5\frac{7}{8}}$.

$$\frac{2\frac{2}{3}}{5\frac{7}{8}}=\frac{\frac{8}{3}}{\frac{47}{8}}=\tfrac{8}{3}\div\tfrac{47}{8}=\tfrac{8}{3}\times\tfrac{8}{47}=\tfrac{64}{141}.$$

(3) Simplify $\dfrac{3x}{x-\frac{1}{4}}$.

$$\frac{3x}{x-\frac{1}{4}}=\frac{3x}{\frac{4x-1}{4}}=\frac{3x}{1}\div\frac{4x-1}{4}=\frac{3x}{1}\times\frac{4}{4x-1}$$

$$=\frac{12x}{4x-1}.$$

It is often shorter to multiply both terms of the fraction by the L. C. D. of the fractions contained in the numerator and denominator.

Thus, in (1), multiply both terms by 6; in (2), both terms by 24; in (3), by 4. The results obtained are $\frac{2}{3}$, $\frac{64}{141}$, $\dfrac{12x}{4x-1}$, respectively.

(4) Simplify $\dfrac{x}{1-\dfrac{x}{1+x+\dfrac{x}{1-x+x^2}}}$.

$$\frac{x}{1-\dfrac{x}{1+x+\dfrac{x}{1-x+x^2}}}=\frac{x}{1-\dfrac{x(1-x+x^2)}{(1+x)(1-x+x^2)+x}}$$

$$=\frac{x}{1-\dfrac{x-x^2+x^3}{1+x+x^3}}$$

$$=\frac{x(1+x+x^3)}{1+x+x^3-(x-x^2+x^3)}$$

$$=\frac{x+x^2+x^4}{1+x^2}.$$

The expression $\dfrac{x}{1+x+\dfrac{x}{1-x+x^2}}$ is reduced to the form $\dfrac{x(1-x+x^2)}{(1+x)(1-x+x^2)+x}$, which $=\dfrac{x-x^2+x^3}{1+x+x^3}$.

The expression $\dfrac{x}{1-\dfrac{x-x^2+x^3}{1+x+x^3}}$ is reduced to the form $\dfrac{x(1+x+x^3)}{1+x+x^3-(x-x^2+x^3)}$, which $=\dfrac{x+x^2+x^4}{1+x^2}$.

Ex. 58.

Simplify:

1. $\dfrac{x-1+\dfrac{6}{x-6}}{x-2+\dfrac{3}{x-6}}$.

2. $\dfrac{3}{x+1}-\dfrac{2x-1}{x^2+\dfrac{x}{2}-\dfrac{1}{2}}$.

3. $\dfrac{x-a}{x-\dfrac{(x-b)(x-c)}{x+a}}$.

4. $\dfrac{\dfrac{1}{x-y}-\dfrac{x}{x^2-y^2}}{\dfrac{x}{xy+y^2}-\dfrac{y}{x^2+xy}}$.

5. $\dfrac{\dfrac{x+1}{x-1}+\dfrac{x-1}{x+1}}{\dfrac{x+1}{x-1}-\dfrac{x-1}{x+1}}$.

6. $1-\dfrac{1}{1+\dfrac{1}{x}}$.

7. $1+\dfrac{x}{1+x+\dfrac{2x^2}{1-x}}$.

8. $\dfrac{1}{1-\dfrac{1}{1+\dfrac{1}{x}}}$.

9. $\dfrac{1}{1+\dfrac{x}{1+x+\dfrac{2x^2}{1-x}}}$.

10. $\dfrac{x}{1+\dfrac{1}{x}}+1-\dfrac{1}{x+1}$.

11. $\dfrac{\dfrac{a+b}{b}+\dfrac{b}{a+b}}{\dfrac{1}{a}+\dfrac{1}{b}}$.

12. $\dfrac{2m-3+\dfrac{1}{m}}{\dfrac{2m-1}{m}}$.

Ex. 59.

Miscellaneous Examples.

1. Simplify $\dfrac{x^4-9x^3+7x^2+9x-8}{x^4+7x^3-9x^2-7x+8}$.

2. Find the value of $\dfrac{a^2+b^2-c^2+2ab}{a^2-b^2-c^2+2bc}$ when $a=4$, $b=\frac{1}{2}$, $c=1$.

3. Find the value of $3a^2+\dfrac{2ab^2}{c}-\dfrac{c^3}{b^2}$ when $a=4$, $b=\frac{1}{2}$, $c=1$.

4. Simplify $\dfrac{2}{(x^2-1)^2}-\dfrac{1}{2x^2-4x+2}-\dfrac{1}{1-x^2}$.

5. Simplify $\left(\dfrac{x}{1+\dfrac{1}{x}}+1-\dfrac{1}{x+1}\right)\div\left(\dfrac{x}{1-\dfrac{1}{x}}-x-\dfrac{1}{x-1}\right)$.

6. Find the value of $\left(\dfrac{x-a}{x-b}\right)^3-\dfrac{x-2a+b}{x+a-2b}$ when $x=\dfrac{a+b}{2}$.

7. Simplify $\left\{\dfrac{a+b}{2(a-b)}-\dfrac{a-b}{2(a+b)}+\dfrac{2b^2}{a^2-b^2}\right\}\dfrac{a-b}{2b}$.

8. Simplify $\left(\dfrac{x^2+y^2}{x^2-y^2}-\dfrac{x^2-y^2}{x^2+y^2}\right)\div\left(\dfrac{x+y}{x-y}-\dfrac{x-y}{x+y}\right)$.

9. Simplify

$$\left(\frac{x^2}{y^2}-1\right)\left(\frac{x}{x-y}-1\right)+\left(\frac{x^3}{y^3}-1\right)\left(\frac{x^2+xy}{x^2+xy+y^2}-1\right).$$

10. Divide $x^3+\frac{1}{x^3}-3\left(\frac{1}{x^2}-x^2\right)+4\left(x+\frac{1}{x}\right)$ by $x+\frac{1}{x}$.

11. Find the value of $\frac{x+2a}{2b-x}+\frac{x-2a}{2b+x}-\frac{4ab}{4b^2-x^2}$ when $x=\frac{ab}{a+b}$.

12. Find the value of $\frac{x+y-1}{x-y+1}$ when $x=\frac{a+1}{ab+1}$ and $y=\frac{ab+a}{ab+1}$.

13. Simplify

$$\frac{1}{a(a-b)(a-c)}+\frac{1}{b(b-c)(b-a)}+\frac{1}{c(c-a)(c-b)}.$$

14. Simplify $\dfrac{\dfrac{m^2+n^2}{n}-m}{\dfrac{1}{n}-\dfrac{1}{m}}\times\dfrac{m^2-n^2}{m^3+n^3}.$

15. Simplify $\dfrac{x-4+\dfrac{6}{x+1}}{x-\dfrac{6}{x-1}}\times\dfrac{1-\dfrac{x+5}{x^2-1}}{(x-1)(x-2)}.$

CHAPTER IX.

Fractional Equations.

TO REDUCE EQUATIONS CONTAINING FRACTIONS.

172. (1) $\frac{x}{2}+\frac{x}{4}=12.$

Multiply both sides by 4, the L. C. M. of the denominators.

Then,
$$\begin{aligned} 2x+x&=48\\ 3x&=48\\ \therefore x&=16. \end{aligned}$$

(2) $\frac{x}{6}-4=24-\frac{x}{8}.$

Multiply both sides by 24, the L. C. M. of the denominators.

Then,
$$\begin{aligned} 4x-96&=576-3x\\ 4x+3x&=576+96\\ 7x&=672\\ \therefore x&=96. \end{aligned}$$

(3) $\frac{x}{3}-\frac{x-1}{11}=x-9.$

Multiply by 33, the L. C. M. of the denominators.

Then,
$$\begin{aligned} 11x-3x+3&=33x-297\\ 11x-3x-33x&=-297-3\\ -25x&=-300\\ \therefore x&=12. \end{aligned}$$

Since the minus sign precedes the second fraction, in removing the denominator, the + (understood) before x, the first term of the numerator, is changed to −, and the − before 1, the second term of the numerator, is changed to +.

173. Therefore, to clear an equation of fractions,
Multiply each term by the L. C. M. *of the denominators.*

If a fraction is preceded by a **minus sign**, *the sign of every term of the numerator must be changed when the denominator is removed.*

Ex. 60.

Solve the equations:

1. $5x - \frac{x+2}{2} = 71.$

2. $x - \frac{3-x}{3} = \frac{17}{3}.$

3. $\frac{5-2x}{4} + 2 = x - \frac{6x-8}{2}.$

4. $\frac{5x}{2} - \frac{5x}{4} = \frac{9}{4} - \frac{3-x}{2}.$

5. $2x - \frac{5x-4}{6} = 7 - \frac{1-2x}{5}.$

6. $\frac{x+2}{2} = \frac{14}{9} - \frac{3+5x}{4}.$

7. $\frac{5x+3}{8} - \frac{3-4x}{3} + \frac{x}{2} = \frac{31}{2} - \frac{9-5x}{6}.$

8. $\frac{10x+3}{3} - \frac{6x-7}{2} = 10(x-1).$

9. $\frac{5x-7}{2} - \frac{2x+7}{3} = 3x - 14.$

10. $\frac{7x+5}{6} - \frac{5x-6}{4} = \frac{8-5x}{12}.$

11. $\frac{x+4}{3} - \frac{x-4}{5} = 2 + \frac{3x-1}{15}.$

12. $\frac{3x+5}{7} - \frac{2x+7}{3} + 10 - \frac{3x}{5} = 0.$

13. $\frac{1}{7}(3x-4) + \frac{1}{3}(5x+3) = 43 - 5x.$

14. $\frac{1}{2}(27-2x) = \frac{9}{2} - \frac{1}{10}(7x-54).$

15. $5x - \{8x - 3[16 - 6x - (4 - 5x)]\} = 6.$

16. $\frac{5x-3}{7} - \frac{9-x}{3} = \frac{5x}{2} + \frac{19}{6}(x-4).$

17. $\frac{2x+7}{7} - \frac{9x-8}{11} = \frac{x-11}{2}.$

18. $\frac{8x-15}{3} - \frac{11x-1}{7} = \frac{7x+2}{13}.$

19. $\frac{7x+9}{8} - \frac{3x+1}{7} = \frac{9x-13}{4} - \frac{249-9x}{14}.$

174. If the denominators contain both simple and compound expressions, it is best to remove the simple expressions first, and then each compound expression in turn. After each multiplication the result should be reduced to the simplest form.

(1) $\frac{8x+5}{14} + \frac{7x-3}{6x+2} = \frac{4x+6}{7}.$

Multiply both sides by 14.

Then, $8x + 5 + \frac{49x - 21}{3x + 1} = 8x + 12.$

Transpose and combine, $\frac{49x-21}{3x+1} = 7.$

Multiply by $3x + 1$, $49x - 21 = 21x + 7$

$$28x = 28$$
$$\therefore x = 1.$$

(2) $\frac{3 - \frac{4x}{9}}{4} = \frac{1}{4} - \frac{\frac{7x}{9} - 3}{10}.$

Simplify the complex fractions by multiplying both terms of each fraction by 9.

Then, $\frac{27 - 4x}{36} = \frac{1}{4} - \frac{7x - 27}{90}.$

Multiply both sides by 180.

$$135 - 20x = 45 - 14x + 54$$
$$-6x = -36$$
$$\therefore x = 6.$$

Ex. 61.

Solve the equations:

1. $\frac{9x+20}{36}=\frac{4(x-3)}{5x-4}+\frac{x}{4}.$

2. $\frac{9(2x-3)}{14}+\frac{11x-1}{3x+1}=\frac{9x+11}{7}.$

3. $\frac{10x+17}{18}-\frac{12x+2}{13x-16}=\frac{5x-4}{9}.$

4. $\frac{6x+13}{15}-\frac{3x+5}{5x-25}=\frac{2x}{5}.$

5. $\frac{18x-22}{39-6x}+2x+\frac{1+16x}{24}=4\frac{5}{12}-\frac{101-64x}{24}.$

6. $\frac{6-5x}{15}-\frac{7-2x^2}{14(x-1)}=\frac{1+3x}{21}-\frac{10x-11}{30}+\frac{1}{105}.$

7. $\frac{9x+5}{14}+\frac{8x-7}{6x+2}=\frac{36x+15}{56}+\frac{41}{56}.$

8. $\frac{6x+7}{15}-\frac{2x-2}{7x-6}=\frac{2x+1}{5}.$

9. $\frac{6x+1}{15}-\frac{2x-4}{7x-16}=\frac{2x-1}{5}.$

10. $\frac{7x-6}{35}-\frac{x-5}{6x-101}=\frac{x}{5}.$

175. Literal equations are equations in which all the numbers are represented by letters; the numbers regarded as known numbers are usually represented by the *first* letters of the alphabet.

(1) $(a-x)(a+x)=2a^2+2ax-x^2.$

Then,
$$a^2-x^2=2a^2+2ax-x^2$$
$$-2ax=a^2$$
$$\therefore x=-\frac{a}{2}.$$

(2) $(x-a)(x-b)-(x-b)(x-c)=2(x-a)(a-c).$

$$(x^2-ax-bx+ab)-(x^2-bx-cx+bc)=2(ax-cx-a^2+ac)$$
$$x^2-ax-bx+ab-x^2+bx+cx-bc=2ax-2cx-2a^2+2ac.$$

That is, $-3ax+3cx=-2a^2+2ac-ab+bc$
$$-3(a-c)x=-2a(a-c)-b(a-c)$$
$$-3x=-2a-b$$
$$\therefore x=\frac{2a+b}{3}.$$

Ex. 62.

Solve the equations:

1. $ax+bc=bx+ac.$

2. $2a-cx=3c-5bx.$

3. $a^2x+bx-c=b^2x+cx-d.$

4. $-ac^2+b^2c+abcx=abc+cmx-ac^2x+b^2c-mc.$

5. $(a+x+b)(a+b-x)=(a+x)(b-x)-ab.$

6. $(a^2+x)^2=x^2+4a^2+a^4.$

7. $(a^2-x)(a^2+x)=a^4+2ax-x^2.$

8. $\dfrac{ax-b}{c}+a=\dfrac{x+ac}{c}.$

9. $\dfrac{a(b^2x+x^3)}{bx}=acx+\dfrac{ax^2}{b}.$

10. $ax-\dfrac{3a-bx}{2}=\dfrac{1}{2}.$

11. $6a-\dfrac{4ax-2b}{3}=x.$

12. $\dfrac{x^2-a}{bx}-\dfrac{a-x}{b}=\dfrac{2x}{b}-\dfrac{a}{x}.$

13. $\dfrac{3}{c}-\dfrac{ab-x^2}{bx}=\dfrac{4x-ac}{cx}.$

14. $am-b-\dfrac{ax}{b}+\dfrac{x}{m}=0.$

Ex. 63.

Solve the equations:

1. $\frac{x-3}{4(x-1)}=\frac{x-5}{6(x-1)}+\frac{1}{9}.$

2. $\frac{7}{x-1}=\frac{6x+1}{x+1}-\frac{3(1+2x^2)}{x^2-1}.$

3. $\frac{1}{2(x-3)}-\frac{1}{3(x-2)}=\frac{x-1}{(x-2)(x-3)}.$

4. $1-\frac{2(2x+3)}{9(7-x)}=\frac{6}{7-x}-\frac{5x+1}{4(7-x)}.$

5. $\frac{17}{x+3}-4=\frac{5(21+2x)}{3x+9}-10.$

6. $\frac{3}{x-1}-\frac{x+1}{x-1}=\frac{x^2}{1-x^2}.$

7. $(x-a)(x-b)=(x-a-b)^2.$

8. $(a-b)(x-c)-(b-c)(x-a)-(c-a)(x-b)=0.$

9. $\frac{x^2-x+1}{x-1}+\frac{x^2+x+1}{x+1}=2x.$

10. $\frac{4}{x+2}+\frac{7}{x+3}=\frac{37}{x^2+5x+6}.$

11. $(x+1)^2=x[6-(1-x)]-2.$

CHAPTER X.

Problems.

Ex. 64.

Ex. Find the number the sum of whose third and fourth parts is equal to 12.

Let $\qquad x =$ the number.

Then $\qquad \frac{x}{3} =$ the third part of the number,

and $\qquad \frac{x}{4} =$ the fourth part of the number.

$\therefore \frac{x}{3} + \frac{x}{4} =$ the sum of the two parts.

But $\qquad 12 =$ the sum of the two parts.

$$\therefore \frac{x}{3} + \frac{x}{4} = 12.$$

Multiply both sides by 12:

$$4x + 3x = 144$$
$$7x = 144.$$
$$\therefore x = 20\tfrac{4}{7}.$$

1. Find the number whose third and fourth parts together make 14.

2. Find the number whose third part exceeds its fourth part by 14.

3. The half, fourth, and fifth of a certain number are together equal to 76; find the number.

4. Find the number whose double exceeds its half by 12.

5. Divide 60 into two such parts that a seventh of one part may be equal to an eighth of the other.

6. Divide 50 into two such parts that a fourth of one part increased by five-sixths of the other part may be equal to 40.

7. Divide 100 into two such parts that a fourth of one part diminished by a third of the other part may be equal to 11.

8. The sum of the fourth, fifth, and sixth parts of a certain number exceeds the half of the number by 112. What is the number?

9. The sum of two numbers is 5760, and their difference is equal to one-third of the greater. What are the numbers?

10. Find a number such that the sum of its fifth and its seventh parts shall exceed the difference of its fourth and its seventh parts by 99.

11. In a mixture of wine and water, the wine was 25 gallons more than half of the mixture, and the water 5 gallons less than one-third of the mixture. How many gallons were there of each?

12. In a certain weight of gunpowder the saltpetre was 6 pounds more than half of the weight, the sulphur 5 pounds less than the third, and the charcoal 3 pounds less than the fourth of the weight. How many pounds were there of each?

13. Divide 46 into two parts such that if one part be divided by 7, and the other by 3, the sum of the quotients shall be 10.

14. A house and garden cost $850, and five times the price of the house was equal to twelve times the price of the garden. What is the price of each?

15. A man leaves the half of his property to his wife, a sixth to each of his two children, a twelfth to his brother, and the remainder, amounting to $600, to his sister. What was the amount of his property?

16. The sum of two numbers is a and their difference is b; find the numbers.

17. Find two numbers of which the sum is 70, such that the first divided by the second gives 2 as a quotient and 1 as a remainder.

18. Find two numbers of which the difference is 25, such that the second divided by the first gives 4 as a quotient and 4 as a remainder.

19. Find four consecutive numbers whose sum is 82.

NOTE I. It is to be remembered that if x represent a person's age at the present time, his age a years ago will be represented by $x - a$, and a years hence by $x + a$.

Ex. In eight years a boy will be three times as old as he was eight years ago. How old is he?

Let $x =$ the number of years of his age.

Then $x - 8 =$ the number of years of his age eight years ago,

and $x + 8 =$ the number of years of his age eight years hence.

$$\begin{aligned} \therefore x + 8 &= 3(x - 8) \\ x + 8 &= 3x - 24 \\ x - 3x &= -24 - 8 \\ -2x &= -32 \\ x &= 16. \end{aligned}$$

20. A is 72 years old, and B's age is two-thirds of A's. How long is it since A was five times as old as B?

21. A mother is 70 years old, her daughter is half that age. How long is it since the mother was three and one-third times as old as the daughter?

22. A father is three times as old as the son; four years ago the father was four times as old as the son then was. What is the age of each?

23. A is twice as old as B, and seven years ago their united ages amounted to as many years as now represent the age of A. Find the ages of A and B.

NOTE II. If A can do a piece of work in x days, the part of the work that he can do in one day will be represented by $\frac{1}{x}$. Thus, if he can do the work in 5 days, in one day he can do $\frac{1}{5}$ of the work.

Ex. A can do a piece of work in 5 days, and B can do it in 4 days. How long will it take A and B together to do the work?

Let $x =$ the number of days it will take A and B together.
Then $\frac{1}{x} =$ the part they can do in one day.
Now $\frac{1}{5} =$ the part A can do in one day,
and $\frac{1}{4} =$ the part B can do in one day.
$\therefore \frac{1}{5} + \frac{1}{4} =$ the part A and B can do in one day.

$$\therefore \frac{1}{5} + \frac{1}{4} = \frac{1}{x}$$
$$4x + 5x = 20$$
$$9x = 20$$
$$x = 2\tfrac{2}{9}$$

Therefore they will do the work in $2\frac{2}{9}$ days.

24. A can do a piece of work in 5 days, B in 6 days, and C in $7\frac{1}{2}$ days; in what time will they do it, all working together?

25. A can do a piece of work in $2\frac{1}{2}$ days, B in $3\frac{1}{3}$ days, and C in $3\frac{3}{4}$ days; in what time will they do it, all working together?

26. Two men who can separately do a piece of work in 15 days and 16 days, can, with the help of another, do it in 6 days. How long would it take the third man to do it alone?

27. A and B together can reap a field in 12 hours, A and C in 16 hours, and A by himself in 20 hours. In what time can B and C together reap it? In what time can A, B, and C together reap it?

28. A and B together can do a piece of work in 12 days, A and C in 15 days, B and C in 20 days. In what time can they do it, all working together?

NOTE III. If a pipe can fill a vessel in x hours, the part of the vessel filled by it in one hour will be represented by $\frac{1}{x}$. Thus, if a pipe will fill a vessel in 3 hours, in 1 hour it will fill $\frac{1}{3}$ of the vessel.

29. A tank can be filled by two pipes in 24 minutes and 30 minutes respectively, and emptied by a third in 20 minutes. In what time will it be filled if all three are running together?

30. A tank can be filled in 15 minutes by two pipes, A and B, running together. After A has been running by itself for 5 minutes, B is also turned on, and the tank is filled in 13 minutes more. In what time may it be filled by each pipe separately?

31. A cistern could be filled by two pipes in 6 hours and 8 hours respectively, and could be emptied by a third in 12 hours. In what time would the cistern be filled if the pipes were all running together?

32. A tank can be filled by three pipes in 1 hour and 20 minutes, 3 hours and 20 minutes, and 5 hours, respectively. In what time will the tank be filled when all three pipes are running together?

NOTE IV. In questions involving distance, time, and rate:

$$\frac{\text{Distance}}{\text{Rate}} = \text{Time}.$$

Thus, if a man travels 40 miles at the rate of 4 miles an hour,

$$\frac{40}{4} = \text{number of hours required.}$$

Ex. A courier who goes at the rate of $31\frac{1}{2}$ miles in 5 hours is followed, after 8 hours, by another who

goes at the rate of $22\frac{1}{2}$ miles in 3 hours. In how many hours will the second overtake the first?

Since the first goes $31\frac{1}{2}$ miles in 5 hours, his rate per hour is $6\frac{3}{10}$ miles.

Since the second goes $22\frac{1}{2}$ miles in 3 hours, his rate per hour is $7\frac{1}{2}$ miles.

Let $x =$ the number of hours the first is travelling.
Then $x - 8 =$ the number of hours the second is travelling.
Then $6\frac{3}{10}x =$ the number of miles the first travels;
$(x-8)7\frac{1}{2} =$ the number of miles the second travels.

They both travel the same distance.

$$\therefore 6\tfrac{3}{10}x = (x-8)7\tfrac{1}{2}.$$

The solution of which gives 42 hours.

33. A sets out and travels at the rate of 7 miles in 5 hours. Eight hours afterwards B sets out from the same place and travels in the same direction, at the rate of 5 miles in 3 hours. In how many hours will B overtake A?

34. A person walks to the top of a mountain at the rate of $2\frac{1}{2}$ miles an hour, and down the same way at the rate of $3\frac{1}{2}$ miles an hour, and is out 5 hours. How far is it to the top of the mountain?

35. The distance between London and Edinburgh is 360 miles. One traveller starts from Edinburgh, and travels at the rate of 10 miles an hour; another starts at the same time from London, and travels at the rate of 8 miles an hour. How far from London will they meet?

36. Two persons set out from the same place in opposite directions. The rate of one of them per hour is a mile less than double that of the other, and in 4 hours they are 32 miles apart. Determine their rates.

37. In going a certain distance, a train travelling 35 miles an hour takes 2 hours less than one travelling 25 miles an hour. Determine the distance.

NOTE V. In problems relating to clocks, it is to be observed that the minute-hand moves *twelve times* as fast as the hour-hand.

Ex. Find the time between two and three o'clock when the hands of a clock are:

I. Together;
II. At right angles to each other;
III. Opposite to each other.

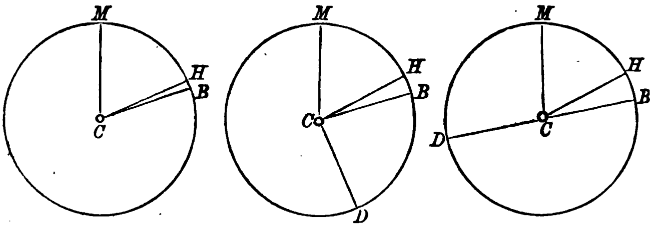


Fig. 1. Fig. 2. Fig. 3.

I. Let CH and CM (Fig. 1) denote the positions of the hour and minute hands at 2 o'clock, and CB the position of both hands when together.

Then arc HB = *one-twelfth* of arc MB.

Let x = number of minute-spaces in arc MB.

Then $\frac{x}{12}$ = number of minute-spaces in arc HB,

and 10 = number of minute-spaces in arc MH.

Now arc MB = arc MH + arc HB.

That is, $x = 10 + \frac{x}{12}$.

The solution of this equation gives $x = 10\frac{10}{11}$.

Hence, the time is $10\frac{10}{11}$ minutes past 2 o'clock.

II. Let CB and CD (Fig. 2) denote the positions of the hour and minute hands when at right angles to each other.

Let x = number of minute-spaces in arc $MHBD$.

Then $\frac{x}{12}$ = number of minute-spaces in arc HB,

and 10 = number of minute-spaces in arc MH,

15 = number of minute-spaces in arc BD.

Now arc $MHBD$ = arcs $MH + HB + BD$.

That is, $x = 10 + \frac{x}{12} + 15.$

The solution of this equation gives $x = 27\frac{3}{11}$.

Hence, the time is $27\frac{3}{11}$ minutes past 2 o'clock.

III. Let CB and CD (Fig. 3) denote the positions of the hour and minute hands when opposite to each other.

Let x = number of minute-spaces in arc $MHBD$.

Then $\frac{x}{12}$ = number of minute-spaces in arc HB,

and 10 = number of minute-spaces in arc MH,

30 = number of minute-spaces in arc BD.

Now arc $MHBD$ = arcs $MH + HB + BD$.

That is, $x = 10 + \frac{x}{12} + 30.$

The solution of this equation gives $x = 43\frac{7}{11}$.

Hence, the time is $43\frac{7}{11}$ minutes past 2 o'clock.

38. At what time are the hands of a watch together:

I. Between 3 and 4?
II. Between 6 and 7?
III. Between 9 and 10?

39. At what time are the hands of a watch at right angles:

I. Between 3 and 4?
II. Between 4 and 5?
III. Between 7 and 8?

40. At what time are the hands of a watch opposite to each other:

I. Between 1 and 2?
II. Between 4 and 5?
III. Between 8 and 9?

NOTE VI. It is to be observed that if a represent the number of feet in the length of a step or leap, and x the number of steps or leaps taken, then ax will represent the number of feet in the distance made.

Ex. A hare takes 4 leaps to a greyhound's 3; but 2 of the greyhound's leaps are equivalent to 3 of the hare's. The hare has a start of 50 leaps. How many leaps must the greyhound take to catch the hare?

Let $3x =$ the number of leaps taken by the greyhound.
Then $4x =$ the number of leaps of the hare in the same time.
Also, let a denote the number of feet in one leap of the hare.
Then $\frac{3a}{2}$ will denote the number of feet in one leap of the greyhound.

That is, $3x \times \frac{3a}{2} =$ the whole distance,

and $(50 + 4x)a =$ the whole distance.

$$\therefore \frac{9ax}{2} = (50 + 4x)a.$$

Divide by a,
$$\frac{9x}{2} = 50 + 4x$$
$$9x = 100 + 8x$$
$$x = 100$$
$$\therefore 3x = 300.$$

Thus the greyhound must take 300 leaps.

41. A hare takes 6 leaps to a dog's 5, and 7 of the dog's leaps are equivalent to 9 of the hare's. The hare has a start of 50 of her own leaps. How many leaps will the hare take before she is caught?

42. A greyhound makes 3 leaps while a hare makes 4; but 2 of the greyhound's leaps are equivalent to 3 of the hare's. The hare has a start of 50 of the greyhound's leaps. How many leaps does each take before the hare is caught?

43. A greyhound makes 2 leaps while a hare makes 3; but 1 leap of the greyhound is equivalent to 2 of the hare's. The hare has a start of 80 of her own leaps. How many leaps will the hare take before she is caught?

NOTE VII. It is to be observed that if the number of units in the breadth and length of a rectangle be represented by x and $x + a$, respectively, then $x(x + a)$ will represent the number of surface units in the rectangle, the unit of surface having the same name as the linear unit in which the sides of the rectangle are expressed.

44. A rectangle whose length is 5 feet more than its breadth would have its area increased by 22 feet if its length and breadth were each made a foot more. Find its dimensions.

45. A rectangle has its length and breadth respectively 5 feet longer and 3 feet shorter than the side of the equivalent square. Find its area.

46. The length of a rectangle is an inch less than double its breadth; and when a strip 3 inches wide is cut off all round, the area is diminished by 210 inches. Find the size of the rectangle at first.

47. The length of a floor exceeds the breadth by 4 feet; if each dimension were increased by 1 foot, the area of the room would be increased by 27 square feet. Find its dimensions.

NOTE VIII. It is to be observed that if b pounds of metal lose a pounds when weighed in water, 1 pound will lose $\frac{1}{b}$ of a pounds, or $\frac{a}{b}$ of a pound.

48. A mass of tin and lead weighing 180 pounds loses 21 pounds when weighed in water; and it is known that 37 pounds of tin lose 5 pounds, and 23 pounds of lead lose 2 pounds, when weighed in water. How many pounds of tin and of lead in the mass?

49. If 19 pounds of gold lose 1 pound, and 10 pounds of silver lose 1 pound, when weighed in water, find the amount of each in a mass of gold and silver weighing 106 pounds in air and 99 pounds in water.

50. There are two silver cups, and one cover for both. The first weighs 12 ounces, and with the cover weighs twice as much as the other without it; but the second with the cover weighs one-third more than the first without it. Find the weight of the cover.

51. A man wishes to enclose a circular piece of ground with palisades, and finds that if he sets them a foot apart, he will have too few by 150; but if he sets them a yard apart, he will have too many by 70. What is the circuit of the piece of ground?

52. A and B shoot by turns at a target. A puts 7 bullets out of 12, and B 9 out of 12, into the centre. Between them they put in 32 bullets. How many shots did each fire?

53. A boy buys a number of apples at the rate of 5 for 2 pence. He sells half of them at 2 a penny and the rest at 3 a penny, and clears a penny by the transaction. How many does he buy?

54. A boy who runs at the rate of 12 yards per second starts 20 yards behind another whose rate is $10\frac{1}{2}$ yards per second. How soon will the first boy be 10 yards ahead of the second?

55. A merchant adds yearly to his capital one-third of it, but takes from it, at the end of each year, $5000 for expenses. At the end of the third year, after deducting the last $5000, he has twice his original capital. How much had he at first?

CHAPTER XI.

SIMULTANEOUS EQUATIONS OF THE FIRST DEGREE.

176. If *one* equation contain *two* unknown quantities, an *indefinite number of pairs* of values may be found that will satisfy the equation.

Thus, in the equation $x + y = 10$, *any values* may be given to x, and *corresponding values* for y may be found. *Any pair* of these values substituted for x and y will satisfy the equation.

177. But if a second equation be given, expressing *different relations* between the unknown quantities, only *one pair* of values of x and y can be found that will satisfy *both* equations.

Thus, if besides the equation $x + y = 10$, another equation, $x - y = 2$, be given, it is evident that the values of x and y which will satisfy both equations are

$$\left.\begin{array}{l} x = 6 \\ y = 4 \end{array}\right\},$$

for $6 + 4 = 10$, and $6 - 4 = 2$; and these are the *only* values of x and y that will satisfy *both* equations.

178. Equations that express *different* relations between the unknown quantities are called **independent equations.**

Thus, $x + y = 10$ and $x - y = 2$ are independent equations; they express *different* relations between x and y. But $x + y = 10$ and $3x + 3y = 30$ are not independent

equations; one is derived immediately from the other, and both express the *same* relation between the unknown quantities.

179. Equations that are to be satisfied by the *same values* of the unknown quantities are called **simultaneous equations.**

180. Simultaneous equations are solved by combining the equations so as to obtain a single equation containing only one unknown quantity; and this process is called **elimination.**

Three methods of elimination are generally given:

I. By Addition or Subtraction.
II. By Substitution.
III. By Comparison.

ELIMINATION BY ADDITION OR SUBTRACTION.

(1) Solve:
$$\left.\begin{aligned} 2x - 3y &= 4 \\ 3x + 2y &= 32 \end{aligned}\right\} \quad \begin{matrix}(1)\\(2)\end{matrix}$$

Multiply (1) by 2 and (2) by 3,
$$4x - 6y = 8 \quad (3)$$
$$9x + 6y = 96 \quad (4)$$
Add (3) and (4), $\quad 13x = 104$
$$\therefore x = 8.$$
Substitute the value of x in (2),
$$24 + 2y = 32.$$
$$\therefore y = 4.$$

In this solution y is eliminated by *addition.*

(2) Solve:
$$\left.\begin{aligned} 6x + 35y &= 177 \\ 8x - 21y &= 33 \end{aligned}\right\} \quad \begin{matrix}(1)\\(2)\end{matrix}$$

Multiply (1) by 4 and (2) by 3,
$$24x + 140y = 708 \quad (3)$$
$$24x - 63y = 99 \quad (4)$$
Subtract (4) from (3), $\quad 203y = 609$
$$\therefore y = 3.$$

Substitute the value of y in (2),

$$8x - 63 = 33.$$

$$\therefore x = 12.$$

In this solution x is eliminated by *subtraction*.

181. Hence, to eliminate an unknown quantity by addition or subtraction,

Multiply the equations by such numbers as will make the coefficients of this unknown quantity equal in the resulting equations.

Add the resulting equations, or subtract one from the other, according as these equal quantities have unlike or like signs.

NOTE. It is generally best to select that unknown quantity to be eliminated which requires the smallest multipliers to make its coefficients equal; and the smallest multiplier for each equation is found by dividing the L. C. M. of the coefficients of this unknown quantity by the given coefficient in that equation. Thus, in example (2), the L. C. M. of 6 and 8 (the coefficients of x), is 24, and hence the smallest multipliers of the two equations are 4 and 3 respectively.

Sometimes the solution is simplified by first adding the given equations, or by subtracting one from the other.

(3)

$$x + 49y = 51 \quad (1)$$

$$49x + y = 99 \quad (2)$$

Add (1) and (2), $\quad 50x + 50y = 150 \quad (3)$

Divide (3) by 50, $\quad x + y = 3. \quad (4)$

Subtract (4) from (1), $\quad 48y = 48.$

$$\therefore y = 1.$$

Subtract (4) from (2), $\quad 48x = 96.$

$$\therefore x = 2.$$

Ex. 65.

Solve by addition or subtraction:

1. $\left.\begin{array}{l} 2x + 3y = 7 \\ 4x - 5y = 3 \end{array}\right\}$ 3. $\left.\begin{array}{l} 7x + 2y = 30 \\ y - 3x = 2 \end{array}\right\}$ 5. $\left.\begin{array}{l} 5x + 4y = 58 \\ 3x + 7y = 67 \end{array}\right\}$

2. $\left.\begin{array}{l} x - 2y = 4 \\ 2x - y = 5 \end{array}\right\}$ 4. $\left.\begin{array}{l} 3x - 5y = 51 \\ 2x + 7y = 3 \end{array}\right\}$ 6. $\left.\begin{array}{l} 3x + 2y = 39 \\ 3y - 2x = 13 \end{array}\right\}$

7. $\left.\begin{aligned} 3x - 4y &= -5 \\ 4x - 5y &= 1 \end{aligned}\right\}$

8. $\left.\begin{aligned} 11x + 3y &= 100 \\ 4x - 7y &= 4 \end{aligned}\right\}$

9. $\left.\begin{aligned} x + 49y &= 693 \\ 49x + y &= 357 \end{aligned}\right\}$

10. $\left.\begin{aligned} 17x + 3y &= 57 \\ 16y - 3x &= 23 \end{aligned}\right\}$

11. $\left.\begin{aligned} 12x + 7y &= 176 \\ 3y - 19x &= 3 \end{aligned}\right\}$

12. $\left.\begin{aligned} 2x - 7y &= 8 \\ 4y - 9x &= 19 \end{aligned}\right\}$

13. $\left.\begin{aligned} 69y - 17x &= 103 \\ 14x - 13y &= -41 \end{aligned}\right\}$

14. $\left.\begin{aligned} 17x + 30y &= 59 \\ 19x + 28y &= 77 \end{aligned}\right\}$

Elimination by Substitution.

(1) Solve: $\left.\begin{aligned} 2x + 3y &= 8 \\ 3x + 7y &= 7 \end{aligned}\right\}$

$$2x + 3y = 8 \qquad (1)$$

$$3x + 7y = 7 \qquad (2)$$

Transpose $3y$ in (1), $2x = 8 - 3y.$ (3)

Divide by coefficient of x, $x = \frac{8 - 3y}{2}$ (4)

Substitute the value of x in (2),

$$3\left(\frac{8 - 3y}{2}\right) + 7y = 7$$

$$\frac{24 - 9y}{2} + 7y = 7$$

$$24 - 9y + 14y = 14$$

$$5y = -10.$$

$$\therefore y = -2.$$

Substitute the value of y in (1),

$$2x - 6 = 8.$$

$$\therefore x = 7.$$

182. Hence, to eliminate an unknown quantity by substitution,

From one of the equations obtain the value of one of the unknown quantities in terms of the other.

Substitute for this unknown quantity its value in the other equation, and reduce the resulting equation.

Ex. 66.

Solve by substitution:

1. $\left.\begin{aligned} 3x-4y&=2 \\ 7x-9y&=7 \end{aligned}\right\}$

2. $\left.\begin{aligned} 7x-5y&=24 \\ 4x-3y&=11 \end{aligned}\right\}$

3. $\left.\begin{aligned} 3x+2y&=32 \\ 20x-3y&=1 \end{aligned}\right\}$

4. $\left.\begin{aligned} 11x-7y&=37 \\ 8x+9y&=41 \end{aligned}\right\}$

5. $\left.\begin{aligned} 7x+5y&=60 \\ 13x-11y&=10 \end{aligned}\right\}$

6. $\left.\begin{aligned} 6x-7y&=42 \\ 7x-6y&=75 \end{aligned}\right\}$

7. $\left.\begin{aligned} 10x+9y&=290 \\ 12x-11y&=130 \end{aligned}\right\}$

8. $\left.\begin{aligned} 3x-4y&=18 \\ 3x+2y&=0 \end{aligned}\right\}$

9. $\left.\begin{aligned} 9x-5y&=52 \\ 8y-3x&=8 \end{aligned}\right\}$

10. $\left.\begin{aligned} 5x-3y&=4 \\ 12y-7x&=10 \end{aligned}\right\}$

11. $\left.\begin{aligned} 9y-7x&=13 \\ 15x-7y&=9 \end{aligned}\right\}$

12. $\left.\begin{aligned} 5x-2y&=51 \\ 19x-3y&=180 \end{aligned}\right\}$

13. $\left.\begin{aligned} 4x+9y&=106 \\ 8x+17y&=198 \end{aligned}\right\}$

14. $\left.\begin{aligned} 8x+3y&=3 \\ 12x+9y&=3 \end{aligned}\right\}$

Elimination by Comparison.

Solve: $\left.\begin{aligned} 2x-9y&=11 \\ 3x-4y&=7 \end{aligned}\right\}$

$$2x-9y=11 \qquad (1)$$
$$3x-4y=7 \qquad (2)$$

Transpose $9y$ in (1) and $4y$ in (2), $2x=11+9y. \qquad (3)$

$$3x=7+4y. \qquad (4)$$

Divide (3) by 2 and (4) by 3, $x=\frac{11+9y}{2}. \qquad (5)$

$$x=\frac{7+4y}{3}. \qquad (6)$$

Equate the values of x, $\frac{11+9y}{2}=\frac{7+4y}{3} \qquad (7)$

Reduce (7), $33 + 27y = 14 + 8y.$

$$19y = -19.$$

$$\therefore y = -1.$$

Substitute the value of y in (1), $2x + 9 = 11.$

$$\therefore x = 1.$$

183. Hence, to eliminate an unknown quantity by comparison,

From each equation obtain the value of one of the unknown quantities in terms of the other.

Form an equation from these equal values, and reduce the equation.

NOTE. If, in the last example, (3) be divided by (4), the resulting equation, $\frac{2}{3} = \frac{11 + 9y}{7 + 4y}$, would, when reduced, give the value of y. This is the shortest method, and therefore to be preferred.

Ex. 67.

Solve by comparison:

1. $\left.\begin{array}{r} x + 15y = 53 \\ 3x + \quad y = 27 \end{array}\right\}$

2. $\left.\begin{array}{r} 4x + \ 9y = 51 \\ 8x - 13y = \ 9 \end{array}\right\}$

3. $\left.\begin{array}{r} 4x + 3y = 48 \\ 5y - 3x = 22 \end{array}\right\}$

4. $\left.\begin{array}{r} 2x + 3y = 43 \\ 10x - \quad y = \ 7 \end{array}\right\}$

5. $\left.\begin{array}{r} 5x - \ 7y = \ 33 \\ 11x + 12y = 100 \end{array}\right\}$

6. $\left.\begin{array}{r} 5x + 7y = 43 \\ 11x + 9y = 69 \end{array}\right\}$

7. $\left.\begin{array}{r} 8x - 21y = \ 33 \\ 6x + 35y = 177 \end{array}\right\}$

8. $\left.\begin{array}{r} 3y - 7x = \ 4 \\ 2y + 5x = 22 \end{array}\right\}$

9. $\left.\begin{array}{r} 21y + 20x = 165 \\ 77y - 30x = 295 \end{array}\right\}$

10. $\left.\begin{array}{r} 11x - 10y = 14 \\ 5x + \ 7y = 41 \end{array}\right\}$

11. $\left.\begin{array}{r} 7y - 3x = 139 \\ 2x + 5y = \ 91 \end{array}\right\}$

12. $\left.\begin{array}{r} 17x + 12y = \ 59 \\ 19x - \ 4y = 153 \end{array}\right\}$

13. $\left.\begin{array}{r} 24x + \ 7y = \ 27 \\ 8x - 33y = 115 \end{array}\right\}$

14. $\left.\begin{array}{r} x = 3y - 19 \\ y = 3x - 23 \end{array}\right\}$

184. Each equation must be simplified, if necessary, before the elimination is performed.

(1) Solve: $\left.\begin{array}{l}(x-1)(y+2)=(x-3)(y-1)+8\\ \dfrac{2x-1}{5}-\dfrac{3(y-2)}{4}=1\end{array}\right\}$

$$(x-1)(y+2)=(x-3)(y-1)+8 \qquad (1)$$
$$\frac{2x-1}{5}-\frac{3(y-2)}{4}=1 \qquad (2)$$

Simplify (1), $xy+2x-y-2=xy-x-3y+3+8.$

Transpose and combine, $3x+2y=13.$ (3)

Simplify (2), $8x-4-15y+30=20.$

Transpose and combine, $8x-15y=-6.$ (4)

Multiply (3) by 8, $24x+16y=104.$ (5)

Multiply (4) by 3, $24x-45y=-18.$ (6)

Subtract (6) from (5), $61y=122.$

$\therefore y=2.$

Substitute the value of y in (3), $3x+4=13.$

$\therefore x=3.$

Ex. 68.

Solve:

1. $\left.\begin{array}{l}x(y+7)=y(x+1)\\ 2x+20=3y+1\end{array}\right\}$

2. $\left.\begin{array}{l}2x-\dfrac{y-3}{5}-4=0\\ 3y+\dfrac{x-2}{3}-9=0\end{array}\right\}$

3. $\left.\begin{array}{l}\dfrac{2}{x+3}=\dfrac{3}{y-2}\\ 5(x+3)=3(y-2)+2\end{array}\right\}$

4. $\left.\begin{array}{l}\dfrac{x-4}{5}-\dfrac{y+2}{10}=0\\ \dfrac{x}{6}+\dfrac{y-2}{4}=3\end{array}\right\}$

5. $\left.\begin{array}{l}(x+1)(y+2)-(x+2)(y+1)=-1\\ 3(x+3)-4(y+4)=-8\end{array}\right\}$

6. $\left.\begin{array}{l}\dfrac{x-2}{5}-\dfrac{10-x}{3}=\dfrac{y-10}{4}\\ \dfrac{2y+4}{3}-\dfrac{2x+y}{8}=\dfrac{x+13}{4}\end{array}\right\}$

7. $\left.\begin{aligned}\frac{x+1}{3}-\frac{y+2}{4}&=\frac{2(x-y)}{5}\\ \frac{x-3}{4}-\frac{y-3}{3}&=2y-x\end{aligned}\right\}$

8. $\left.\begin{aligned}\frac{x+y}{y-x}&=\frac{15}{8}\\ 9x-\frac{3y+44}{7}&=100\end{aligned}\right\}$

9. $\left.\begin{aligned}\frac{3x-5y}{2}+3&=\frac{2x+y}{5}\\ 8-\frac{x-2y}{4}&=\frac{x}{2}+\frac{y}{3}\end{aligned}\right\}$

10. $\left.\begin{aligned}\frac{x-4}{5}&=\frac{y+2}{10}\\ \frac{x}{6}+\frac{y-2}{4}&=3\end{aligned}\right\}$

11. $\left.\begin{aligned}\frac{3x+12y}{11}&=9\\ \frac{1-3x}{7}&=\frac{11-3y}{5}\end{aligned}\right\}$

12. $\left.\begin{aligned}5x-\tfrac{1}{4}(5y+2)&=32\\ 3y+\tfrac{1}{3}(x+2)&=9\end{aligned}\right\}$

13. $\left.\begin{aligned}7(x-1)&=3(y+8)\\ \frac{4x+2}{9}&=\frac{5y+9}{2}\end{aligned}\right\}$

14. $\left.\begin{aligned}\frac{5x-6y}{13}+3x&=4y-2\\ \frac{5x+6y}{6}-\frac{3x-2y}{4}&=2y-2\end{aligned}\right\}$

Literal Simultaneous Equations.

185. The method of solving literal simultaneous equations is as follows:

Solve: $\left.\begin{aligned}ax+by&=m\\ cx+dy&=n\end{aligned}\right\}$

$$ax+by=m \quad (1)$$
$$cx+dy=n \quad (2)$$

Multiply (1) by c, $acx+bcy=cm$ (3)

Multiply (2) by a, $acx+ady=an$ (4)

Subtract (4) from (3), $(bc-ad)y=cm-an$

Divide by coefficient of y, $y=\dfrac{cm-an}{bc-ad}$.

To find the value of x:

Multiply (1) by d, $\quad adx + bdy = dm \quad (5)$

Multiply (2) by b, $\quad bcx + bdy = bn \quad (6)$

Subtract (6) from (5), $\quad (ad - bc)x = dm - bn$

Divide by coefficient of x, $\quad x = \dfrac{dm - bn}{ad - bc}$.

Ex. 69.

Solve:

1. $\left.\begin{aligned} x + y &= a \\ x - y &= b \end{aligned}\right\}$

2. $\left.\begin{aligned} ax + by &= c \\ px + qy &= r \end{aligned}\right\}$

3. $\left.\begin{aligned} mx + ny &= a \\ px + qy &= b \end{aligned}\right\}$

4. $\left.\begin{aligned} ax + by &= e \\ ax + cy &= d \end{aligned}\right\}$

5. $\left.\begin{aligned} mx - ny &= r \\ m'x + n'y &= r' \end{aligned}\right\}$

6. $\left.\begin{aligned} ax + by &= c \\ dx + fy &= c^2 \end{aligned}\right\}$

7. $\left.\begin{aligned} \frac{x}{a} + \frac{y}{b} &= c \\ \frac{x}{b} + \frac{y}{a} &= -c \end{aligned}\right\}$

8. $\left.\begin{aligned} abx + cdy &= 2 \\ ax - cy &= \frac{d - b}{bd} \end{aligned}\right\}$

9. $\left.\begin{aligned} \frac{a}{b + y} &= \frac{b}{3a + x} \\ ax + 2by &= d \end{aligned}\right\}$

10. $\left.\begin{aligned} \frac{x}{a + b} - \frac{y}{a - b} &= \frac{1}{a + b} \\ \frac{x}{a + b} + \frac{y}{a - b} &= \frac{1}{a - b} \end{aligned}\right\}$

186. Fractional simultaneous equations, of which the denominators are simple expressions and contain the unknown quantities, may be solved as follows:

(1) Solve: $\quad \left.\begin{aligned} \frac{a}{x} + \frac{b}{y} &= m \\ \frac{c}{x} + \frac{d}{y} &= n \end{aligned}\right\}$

$$\frac{a}{x} + \frac{b}{y} = m \quad (1)$$

$$\frac{c}{x} + \frac{d}{y} = n \quad (2)$$

Multiply (1) by c, $\frac{ac}{x}+\frac{bc}{y}=cm.$ (3)

Multiply (2) by a, $\frac{ac}{x}+\frac{ad}{y}=an.$ (4)

Subtract (4) from (3), $\frac{bc-ad}{y}=cm-an.$

Multiply both sides by y, $bc-ad=(cm-an)y.$

$$\therefore y=\frac{bc-ad}{cm-an}.$$

Multiply (1) by d, $\frac{ad}{x}+\frac{bd}{y}=dm.$ (5)

Multiply (2) by b, $\frac{bc}{x}+\frac{bd}{y}=bn.$ (6)

Subtract (6) from (5), $\frac{ad-bc}{x}=dm-bn.$

Multiply both sides by x, $ad-bc=(dm-bn)x.$

$$\therefore x=\frac{ad-bc}{dm-bn}.$$

(2) Solve:

$$\left.\begin{aligned}\frac{5}{3x}+\frac{2}{5y}&=7\\ \frac{7}{6x}-\frac{1}{10y}&=3\end{aligned}\right\}$$

$$\frac{5}{3x}+\frac{2}{5y}=7 \qquad (1)$$

$$\frac{7}{6x}-\frac{1}{10y}=3 \qquad (2)$$

Multiply (2) by 4, $\frac{14}{3x}-\frac{2}{5y}=12.$ (3)

Add (1) and (3), $\frac{19}{3x}=19.$

Divide both sides by 19, $\frac{1}{3x}=1.$

$$\therefore x=\tfrac{1}{3}.$$

Substitute the value of x in (1),

$$5+\frac{2}{5y}=7.$$

Transpose, $\frac{2}{5y}=2.$

Divide both sides by 2, $\frac{1}{5y}=1.$

$$\therefore y=\tfrac{1}{5}.$$

Ex. 70.

Solve:

1. $\left.\begin{array}{l}\dfrac{1}{x}+\dfrac{2}{y}=10\\[2mm]\dfrac{4}{x}+\dfrac{3}{y}=20\end{array}\right\}$

2. $\left.\begin{array}{l}\dfrac{1}{x}+\dfrac{2}{y}=a\\[2mm]\dfrac{3}{x}+\dfrac{4}{y}=b\end{array}\right\}$

3. $\left.\begin{array}{l}\dfrac{2}{x}-\dfrac{5}{3y}=\dfrac{4}{27}\\[2mm]\dfrac{1}{4x}+\dfrac{1}{y}=\dfrac{11}{72}\end{array}\right\}$

4. $\left.\begin{array}{l}\dfrac{1}{x}+\dfrac{2}{y}=4\\[2mm]\dfrac{3}{x}-\dfrac{2}{y}=4\end{array}\right\}$

5. $\left.\begin{array}{l}\dfrac{3}{x}-\dfrac{4}{y}=5\\[2mm]\dfrac{4}{x}-\dfrac{5}{y}=6\end{array}\right\}$

6. $\left.\begin{array}{l}\dfrac{a}{x}+\dfrac{b}{y}=\dfrac{ac}{b}\\[2mm]\dfrac{b}{x}+\dfrac{a}{y}=\dfrac{bc}{a}\end{array}\right\}$

7. $\left.\begin{array}{l}\dfrac{2}{ax}+\dfrac{3}{by}=5\\[2mm]\dfrac{5}{ax}-\dfrac{2}{by}=3\end{array}\right\}$

8. $\left.\begin{array}{l}\dfrac{m}{nx}+\dfrac{n}{my}=m+n\\[2mm]\dfrac{n}{x}+\dfrac{m}{y}=m^2+n^2\end{array}\right\}$

9. $\left.\begin{array}{l}\dfrac{a}{x}+\dfrac{b}{y}=m\\[2mm]\dfrac{b}{x}-\dfrac{a}{y}=n\end{array}\right\}$

187. If three simultaneous equations are given, involving three unknown quantities, one of the unknown quantities must be eliminated between *two pairs* of the equations; then a second between the resulting equations.

188. Likewise, if four or more equations are given, involving four or more unknown quantities, one of the unknown quantities must be eliminated between three or more pairs of the equations; then a second between the pairs that can be found of the resulting equations; and so on.

Solve:
$$\left.\begin{array}{r}2x-3y+4z=4\\3x+5y-7z=12\\5x-y-8z=5\end{array}\right\}\quad\begin{array}{r}(1)\\(2)\\(3)\end{array}$$

Eliminate z between two pairs of these equations.

Multiply (1) by 2, $\quad 4x-6y+8z=8 \qquad (4)$

(3) is $\quad 5x-y-8z=5$

Add, $\quad 9x-7y=13 \qquad (5)$

Multiply (1) by 7,	$14x - 21y + 28z = 28$	
Multiply (2) by 4,	$12x + 20y - 28z = 48$	
Add,	$26x - y = 76$	(6)
Multiply (6) by 7,	$182x - 7y = 532$	(7)
(5) is	$9x - 7y = 13$	
Subtract (5) from (7),	$173x = 519$	
	$\therefore x = 3.$	
Substitute the value of x in (6),	$78 - y = 76.$	
	$\therefore y = 2.$	
Substitute the values of x and y in (1),	$6 - 6 + 4z = 4.$	
	$\therefore z = 1.$	

Ex. 71.

Solve:

1. $\left.\begin{array}{l} 5x+3y-6z=4 \\ 3x-y+2z=8 \\ x-2y+2z=2 \end{array}\right\}$

2. $\left.\begin{array}{l} 4x-5y+2z=6 \\ 2x+3y-z=20 \\ 7x-4y+3z=35 \end{array}\right\}$

3. $\left.\begin{array}{l} x+y+z=6 \\ 5x+4y+3z=22 \\ 15x+10y+6z=53 \end{array}\right\}$

4. $\left.\begin{array}{l} 4x-3y+z=9 \\ 9x+y-5z=16 \\ x-4y+3z=2 \end{array}\right\}$

5. $\left.\begin{array}{l} 8x+4y-3z=6 \\ x+3y-z=7 \\ 4x-5y+4z=8 \end{array}\right\}$

6. $\left.\begin{array}{l} 12x+5y-4z=29 \\ 13x-2y+5z=58 \\ 17x-y-z=15 \end{array}\right\}$

7. $\left.\begin{array}{l} y-x+z=-5 \\ z-y-x=-25 \\ x+y+z=35 \end{array}\right\}$

8. $\left.\begin{array}{l} x+y+z=30 \\ 8x+4y+2z=50 \\ 27x+9y+3z=64 \end{array}\right\}$

9. $\left.\begin{array}{l} 15y=24z-10x+41 \\ 15x=12y-16z+10 \\ 18x-(7z-13)=14y \end{array}\right\}$

10. $\left.\begin{array}{l} 3x-y+z=17 \\ 5x+3y-2z=10 \\ 7x+4y-5z=3 \end{array}\right\}$

11. $\left.\begin{array}{l} x+y+z=5 \\ 3x-5y+7z=75 \\ 9x-11z+10=0 \end{array}\right\}$

12. $\left.\begin{array}{l} x+2y+3z=6 \\ 2x+4y+2z=8 \\ 3x+2y+8z=101 \end{array}\right\}$

13. $$\left.\begin{aligned} x-3y-2z&=1\\ 2x-3y+5z&=-19\\ 5x+2y-z&=12 \end{aligned}\right\}$$

14. $$\left.\begin{aligned} 3x-2y&=5\\ 4x-3y+2z&=11\\ x-2y-5z&=-7 \end{aligned}\right\}$$

15. $$\left.\begin{aligned} x+y&=1\\ y+z&=9\\ x+z&=5 \end{aligned}\right\}$$

16. $$\left.\begin{aligned} 2x-3y&=3\\ 3y-4z&=7\\ 4z-5x&=2 \end{aligned}\right\}$$

17. $$\left.\begin{aligned} 3x-4y+6z&=1\\ 2x+2y-z&=1\\ 7x-6y+7z&=2 \end{aligned}\right\}$$

18. $$\left.\begin{aligned} 7x-3y&=30\\ 9y-5z&=34\\ x+y+z&=33 \end{aligned}\right\}$$

19. $$\left.\begin{aligned} x+\frac{y}{2}+\frac{z}{3}&=6\\ y+\frac{z}{2}+\frac{x}{3}&=-1\\ z+\frac{x}{2}+\frac{y}{3}&=17 \end{aligned}\right\}$$

20. $$\left.\begin{aligned} \frac{1}{x}+\frac{2}{y}&=5\\ \frac{3}{y}-\frac{4}{z}&=-6\\ \frac{3}{z}-\frac{4}{x}&=5 \end{aligned}\right\}$$

21. $$\left.\begin{aligned} \frac{1}{x}+\frac{1}{y}-\frac{1}{z}&=a\\ \frac{1}{x}-\frac{1}{y}+\frac{1}{z}&=b\\ \frac{1}{y}+\frac{1}{z}-\frac{1}{x}&=c \end{aligned}\right\}*$$

22. $$\left.\begin{aligned} \frac{2}{x}-\frac{3}{y}+\frac{4}{z}&=2.9\\ \frac{5}{x}-\frac{6}{y}-\frac{7}{z}&=-10.4\\ \frac{9}{y}+\frac{10}{z}-\frac{8}{x}&=14.9 \end{aligned}\right\}$$

23. $$\left.\begin{aligned} \frac{2}{x}+\frac{1}{y}-\frac{3}{z}&=0\\ \frac{3}{z}-\frac{2}{y}-2&=0\\ \frac{1}{x}+\frac{1}{z}-\frac{4}{3}&=0 \end{aligned}\right\}$$

* Subtract from the sum of the three equations each equation separately.

CHAPTER XII.

PROBLEMS PRODUCING SIMULTANEOUS EQUATIONS.

189. It is often necessary in the solution of problems to employ two or more letters to represent the quantities to be found. In all cases the conditions must be sufficient to give just as many equations as there are unknown quantities employed.

If there be *more* equations than unknown quantities, some of them are superfluous or contradictory; if there be *less* equations than unknown quantities, the problem is indeterminate or impossible.

(1) When the greater of two numbers is divided by the less, the quotient is 4 and the remainder 3; and when the sum of the two numbers is increased by 38, and the result divided by the greater of the two numbers, the quotient is 2 and the remainder 2. Find the numbers.

Let $x =$ the greater number,

and $y =$ the smaller number.

Then $$\frac{x-3}{y} = 4,$$

and $$\frac{x+y+38-2}{x} = 2.$$

From the solution of these equations, $x = 47$ and $y = 11$.

(2) If A give B \$10, B will have three times as much money as A. If B give A \$10, A will have twice as much money as B. How much has each?

Let $x =$ number of dollars A has,

and $y =$ number of dollars B has.

Then $y + 10 =$ number of dollars B has,

and $x - 10 =$ number of dollars A has after A gives \$10 to B.

$$\therefore y + 10 = 3(x - 10), \text{ and } x + 10 = 2(y - 10).$$

From the solution of these equations, $x = 22$ and $y = 26$.

Therefore A has \$22 and B \$26.

Ex. 72.

1. The sum of two numbers divided by 2 gives as a quotient 24, and the difference between them divided by 2 gives as a quotient 17. What are the numbers?

2. Three times the greater of two numbers exceeds twice the less by 10; and twice the greater together with three times the less is 24. Find the numbers.

3. Seven years ago the age of a father was four times that of his son; seven years hence the age of the father will be double that of the son. What are their ages?

4. If B give A \$25, they will have equal sums of money; but if A give B \$22, B's money will be double that of A. How much has each?

5. A farmer sold to one person 30 bushels of wheat and 40 bushels of barley for \$67.50; to another person he sold 50 bushels of wheat and 30 bushels of barley for \$85. What was the price of the wheat and of the barley per bushel?

6. If A give B \$5, he will then have \$6 less than B; but if he receive \$5 from B, three times his money will be \$20 more than four times B's. How much has each?

7. The cost of 12 horses and 14 cows is \$1900, the cost of 5 horses and 3 cows is \$650. What is the cost of a horse and a cow respectively?

NOTE I. A fraction of which the terms are unknown may be represented by $\frac{x}{y}$.

Ex. A certain fraction becomes equal to $\frac{1}{2}$ if 3 be added to its numerator, and equal to $\frac{2}{7}$ if 3 be added to its denominator. Determine the fraction.

Let $\frac{x}{y}$ = the required fraction.

By the conditions, $$\frac{x+3}{y}=\frac{1}{2},$$

and $$\frac{x}{y+3}=\frac{2}{7}.$$

From the solution of these equations it is found that

$$x=6,$$
$$y=18.$$

Therefore the fraction $=\frac{6}{18}$.

8. A certain fraction becomes equal to 2 when 7 is added to its numerator, and equal to 1 when 1 is subtracted from its denominator. Determine the fraction.

9. A certain fraction becomes equal to $\frac{1}{2}$ when 7 is added to its denominator, and equal to 2 when 13 is added to its numerator. Determine the fraction.

10. A certain fraction becomes equal to $\frac{7}{9}$ when the denominator is increased by 4, and equal to $\frac{20}{41}$ when the numerator is diminished by 15. Determine the fraction.

11. A certain fraction becomes equal to $\frac{2}{3}$ if 7 be added to the numerator, and equal to $\frac{3}{5}$ if 7 be subtracted from the denominator. Determine the fraction.

12. Find two fractions with numerators 2 and 5 respectively, whose sum is $1\frac{1}{2}$, and if their denominators are interchanged, their sum is 2.

NOTE II. A number consisting of *two* digits which are unknown may be represented by $10x+y$, in which x and y represent the digits of the number. Likewise, a number consisting of *three* digits which

are unknown may be represented by $100x + 10y + z$, in which x, y, and z represent the digits of the number.

For example, consider any number expressed by three digits, as 364. The expression 364 means $300 + 60 + 4$; or, 100 *times* $3 + 10$ *times* $6 + 4$.

Ex. The sum of the two digits of a number is 8, and if 36 be added to the number, the digits will be interchanged. What is the number?

Let $x =$ the digit in the tens' place,
and $y =$ the digit in the units' place.
Then $10x + y =$ the number.
By the conditions, $x + y = 8$, (1)
and $10x + y + 36 = 10y + x$. (2)
From (2), $9x - 9y = -36$.
Divide by 9, $x - y = -4$.
Add (1) and (3), $2x = 4$.
$\therefore x = 2$.
Subtract (3) from (1), $2y = 12$.
$\therefore y = 6$.

Hence the number is 26.

13. The sum of the two digits of a number is 10, and if 54 be added to the number, the digits will be interchanged. What is the number?

14. The sum of the two digits of a number is 6, and if the number be divided by the sum of the digits, the quotient is 4. What is the number?

15. A certain number is expressed by two digits, of which the first is the greater. If the number be divided by the sum of its digits, the quotient is 7; if the digits be interchanged, and the resulting number diminished by 12 be divided by the difference between the two digits, the quotient is 9. What is the number?

16. If a certain number be divided by the sum of its two digits, the quotient is 6 and the remainder 3; if the digits be interchanged, and the resulting number be divided by the sum of the digits, the quotient is 4 and the remainder 9. What is the number?

17. If a certain number be divided by the sum of its two digits diminished by 2, the quotient is 5 and the remainder 1; if the digits be interchanged, and the resulting number be divided by the sum of the digits increased by 2, the quotient is 5 and the remainder 8. Find the number.

18. The first of the two digits of a number is, when doubled, 3 more than the second, and the number itself is less by 6 than five times the sum of the digits. What is the number?

19. A number is expressed by three digits, of which the first and last are alike. By interchanging the digits in the units' and tens' places the number is increased by 54; but if the digits in the tens' and hundreds' places are interchanged, 9 must be added to four times the resulting number to make it equal to the original number. What is the number?

20. A number is expressed by three digits. The sum of the digits is 21; the sum of the first and second exceeds the third by 3; and if 198 be added to the number, the digits in the units' and hundreds' places will be interchanged. Find the number.

21. A number is expressed by three digits. The sum of the digits is 9; the number is equal to forty-two times the sum of the first and second digits; and the third digit is twice the sum of the other two. Find the number.

22. A certain number, expressed by three digits, is equal to forty-eight times the sum of its digits. If 198 be subtracted from the number, the digits in the units' and hundreds' places will be interchanged; and the sum of the extreme digits is equal to twice the middle digit. Find the number.

NOTE III. If a boat move at the rate of x miles an hour in still water, and if it be on a stream that runs at the rate of y miles an hour, then

$x + y$ represents its rate *down* the stream,
$x - y$ represents its rate *up* the stream.

23. A waterman rows 30 miles and back in 12 hours. He finds that he can row 5 miles with the stream in the same time as 3 against it. Find the time he was rowing up and down respectively.

24. A crew, which can pull at the rate of 12 miles an hour down the stream, finds that it takes twice as long to come up the river as to go down. At what rate does the stream flow?

25. A man sculls down a stream, which runs at the rate of 4 miles an hour, for a certain distance in 1 hour and 40 minutes. In returning it takes him 4 hours and 15 minutes to arrive at a point 3 miles short of his starting-place. Find the distance he pulled down the stream and the rate of his pulling.

26. A person rows down a stream a distance of 20 miles and back again in 10 hours. He finds he can row 2 miles against the stream in the same time he can row 3 miles with it. Find the time of his rowing down and of his rowing up the stream; and also the rate of the stream.

NOTE IV. When commodities are mixed, it is to be observed that the quantity of the mixture = the quantity of the ingredients; the cost of the mixture = the cost of the ingredients.

Ex. A wine-merchant has two kinds of wine, which cost 72 cents and 40 cents a quart respectively. How much of each must he take to make a mixture of 50 quarts worth 60 cents a quart?

Let x = required number of quarts worth 72 cents a quart,

and y = required number of quarts worth 40 cents a quart.

Then, $72x$ = cost in cents of the first kind,

$40y$ = cost in cents of the second kind,

and 3000 = cost in cents of the mixture.

$$\therefore x + y = 50,$$

$$72x + 40y = 3000.$$

From which equations the values of x and y may be found.

27. A grocer mixed tea that cost him 42 cents a pound with tea that cost him 54 cents a pound. He had 30 pounds of the mixture, and by selling it at the rate of 60 cents a pound, he gained as much as 10 pounds of the cheaper tea cost him. How many pounds of each did he put into the mixture?

28. A grocer mixes tea that cost him 90 cents a pound with tea that cost him 28 cents a pound. The cost of the mixture is \$61.20. He sells the mixture at 50 cents a pound, and gains \$3.80. How many pounds of each did he put into the mixture?

29. A farmer has 28 bushels of barley worth 84 cents a bushel. With his barley he wishes to mix rye worth \$1.08 a bushel, and wheat worth \$1.44 a bushel, so that the mixture may be 100 bushels, and be worth \$1.20 a bushel. How many bushels of rye and of wheat must he take?

NOTE V. It is to be remembered that if a person can do a piece of work in x days, *the part* of the work he can do in *one* day will be represented by $\frac{1}{x}$.

Ex. A and B together can do a piece of work in 48 days; A and C together can do it in 30 days; B and C together can do it in $26\frac{2}{3}$ days. How long will it take each to do the work?

Let x = the number of days it will take A alone to do the work,
y = the number of days it will take B alone to do the work,
and z = the number of days it will take C alone to do the work.

Then, $\frac{1}{x}$, $\frac{1}{y}$, $\frac{1}{z}$, respectively, will denote the part each can do in a day,

and $\frac{1}{x}+\frac{1}{y}$ will denote the part A and B together can do in a day,

but $\frac{1}{48}$ will denote the part A and B together can do in a day.

Therefore, $$\frac{1}{x}+\frac{1}{y}=\frac{1}{48} \qquad (1)$$

Likewise, $$\frac{1}{x}+\frac{1}{z}=\frac{1}{30} \qquad (2)$$

and $$\frac{1}{y}+\frac{1}{z}=\frac{1}{26\frac{2}{3}}=\frac{3}{80} \qquad (3)$$

Add (1), (2), and (3), $$\frac{2}{x}+\frac{2}{y}+\frac{2}{z}=\frac{11}{120} \qquad (4)$$

Multiply (1) by 2, $$\frac{2}{x}+\frac{2}{y}=\frac{1}{24} \qquad (5)$$

Subtract (5) from (4), $$\frac{2}{z}=\frac{1}{20}$$
$$\therefore z=40.$$

Subtract the double of (2) from (4), $$\frac{2}{y}=\frac{1}{40}.$$
$$\therefore y=80.$$

Subtract the double of (3) from (4), $$\frac{2}{x}=\frac{1}{60}.$$
$$\therefore x=120.$$

30. A and B together earn \$40 in 6 days; A and C together earn \$54 in 9 days; B and C together earn \$80 in 15 days. What does each earn a day?

31. A cistern has three pipes, A, B, and C. A and B will fill it in 1 hour and 10 minutes; A and C in 1 hour and 24 minutes; B and C in 2 hours and 20 minutes. How long will it take each to fill it?

32. A warehouse will hold 24 boxes and 20 bales; 6 boxes and 14 bales will fill half of it. How many of each alone will it hold?

33. Two workmen together complete some work in 20 days; but if the first had worked twice as fast, and the second half as fast, they would have finished it in 15 days. How long would it take each alone to do the work?

34. A purse holds 19 crowns and 6 guineas; 4 crowns and 5 guineas fill $\frac{17}{63}$ of it. How many of each alone will it hold?

35. A piece of work can be completed by A, B, and C together in 10 days; by A and B together in 12 days; by B and C, if B work 15 days and C 30 days. How long will it take each alone to do the work?

36. A cistern has three pipes, A, B, and C. A and B will fill it in a minutes; A and C in b minutes; B and C in c minutes. How long will it take each alone to fill it?

NOTE VI. In considering *the rate of increase or decrease* in quantities, it is usual to take 100 as a *common standard of reference*, so that the increase or decrease is calculated for every 100, and therefore called *per cent.*

It is to be observed that the representative of the number resulting after an increase has taken place is 100 + increase per cent; and after a decrease, 100 − decrease per cent.

Interest depends upon the *time* for which the money is lent, as well as upon the *rate per cent* charged; the rate per cent charged being the rate per cent on the principal for *one year*. Hence,

$$\text{Simple interest} = \frac{\text{Principal} \times \text{Rate} \times \text{Time}}{100},$$

where Time means *number of years or fraction of a year.*

$$\text{Amount} = \text{Principal} + \text{Interest.}$$

In questions relating to stocks, 100 is taken as the representative of the *stock*, the *price* represents its market value, and the *per cent* represents the *interest* which the *stock* bears. Thus, if six per cent stocks are quoted at 108, the meaning is, that the price of \$100 of the stock is \$108, and that the interest derived from \$100 of the *stock* will be $\frac{6}{100}$ of \$100, that is, \$6 a year. The rate of interest on the *money invested* will be $\frac{100}{108}$ of 6 per cent.

37. A man has \$10,000 invested. For a part of this sum he receives 5 per cent interest, and for the rest 4 per cent; the income from his 5 per cent investment is \$50 more than from his 4 per cent. How much has he in each investment?

38. A sum of money, at simple interest, amounted in 6 years to \$26,000, and in 10 years to \$30,000. Find the sum and the rate of interest.

39. A sum of money, at simple interest, amounted in 10 months to \$26,250, and in 18 months to \$27,250. Find the sum and the rate of interest.

40. A sum of money, at simple interest, amounted in m years to a dollars, and in n years to b dollars. Find the sum and the rate of interest.

41. A sum of money, at simple interest, amounted in a months to c dollars, and in b months to d dollars. Find the sum and the rate of interest.

42. A person has a certain capital invested at a certain rate per cent. Another person has \$1000 more capital, and his capital invested at one per cent better than the first, and receives an income \$80 greater. A third person has \$1500 more capital, and his capital invested at two per cent better than the first, and receives an income \$150 greater. Find the capital of each, and the rate at which it is invested.

NOTE VII. If x represent the number of linear units in the length, and y in the width, of a rectangle, xy will represent the number of its units of surface; the surface unit having the same name as the linear unit of its sides.

43. If the sides of a rectangular field were each increased by 2 yards, the area would be increased by 220 square yards; if the length were increased and the breadth were diminished each by 5 yards, the area would be diminished by 185 square yards. What is its area?

44. If a given rectangular floor had been 3 feet longer and 2 feet broader, it would have contained 64 square feet more; but if it had been 2 feet longer and 3 feet broader, it would have contained 68 square feet more. Find the length and breadth of the floor.

45. In a certain rectangular garden there is a strawberry-bed whose sides are one-third of the lengths of the corresponding sides of the garden. The perimeter of the garden exceeds that of the bed by 200 yards; and if the greater side of the garden be increased by 3, and the other by 5 yards, the garden will be enlarged by 645 square yards. Find the length and breadth of the garden.

CHAPTER XIII.

Involution and Evolution.

190. The operation of raising an expression to any required *power* is called **Involution.**

Every case of involution is merely an example of *multiplication*, in which the factors are *equal*. Thus,

$$(2a^3)^2 = 2a^3 \times 2a^3 = 4a^6.$$

191. A power of a simple expression is found by multiplying the exponent of each factor by the exponent of the required factor, and taking the product of the resulting factors. The proof of the law of exponents, in its general form, is:

$$\begin{aligned}(a^m)^n &= a^m \times a^m \times a^m \times \text{..... to } n \text{ factors}\\ &= a^{m+m+m+\text{..... to } n \text{ terms}}\\ &= a^{mn}.\end{aligned}$$

Hence, if the exponent of the required power be a composite number, it may be resolved into prime factors, the power denoted by one of these factors may be found, and the result raised to a power denoted by another, and so on. Thus, the fourth power may be obtained by taking the second power of the second power; the sixth by taking the second power of the third power; the eighth by taking the second power of the second power of the second power.

192. From the **Law of Signs** in multiplication it is evident that,

I. All *even* powers of a number are *positive.*

II. All *odd* powers of a number have the *same sign* as the number itself.

Hence, no *even* power of *any* number can be *negative;* and of two compound expressions whose terms are identical but have opposite signs, the even powers are the same. Thus,

$$(b-a)^2 = \{-(a-b)\}^2 = (a-b)^2.$$

193. A method has been given, § 83, of finding, without actual multiplication, the powers of binomials which have the form $(a \pm b)$.

The same method may be employed when the terms of a binomial have *coefficients* or *exponents.*

(1) $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3.$

(2) $(5x^2 - 2y^3)^3$

$$= (5x^2)^3 - 3(5x^2)^2(2y^3) + 3(5x^2)(2y^3)^2 - (2y^3)^3$$
$$= 125x^6 - 150x^4y^3 + 60x^2y^6 - 8y^9.$$

(3) $(a-b)^4 = a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4.$

(4) $(x^2 - \frac{1}{2}y)^4$

$$= (x^2)^4 - 4(x^2)^3(\tfrac{1}{2}y) + 6(x^2)^2(\tfrac{1}{2}y)^2 - 4x^2(\tfrac{1}{2}y)^3 + (\tfrac{1}{2}y)^4$$
$$= x^8 - 2x^6y + \tfrac{3}{2}x^4y^2 - \tfrac{1}{2}x^2y^3 + \tfrac{1}{16}y^4.$$

194. In like manner, a *polynomial* of three or more terms may be raised to any power by enclosing its terms in parentheses, so as to give the expression the form of a binomial. Thus,

(1) $(a+b+c)^3 = \{a + (b+c)\}^3$

$$= a^3 + 3a^2(b+c) + 3a(b+c)^2 + (b+c)^3$$
$$= a^3 + 3a^2b + 3a^2c + 3ab^2 + 6abc$$
$$+ 3ac^2 + b^3 + 3b^2c + 3bc^2 + c^3.$$

(2) $(x^3-2x^2+3x+4)^2$

$$=\{(x^3-2x^2)+(3x+4)\}^2$$
$$=(x^3-2x^2)^2+2(x^3-2x^2)(3x+4)+(3x+4)^2$$
$$=x^6-4x^5+4x^4+6x^4-4x^3-16x^2+9x^2+24x+16$$
$$=x^6-4x^5+10x^4-4x^3-7x^2+24x+16.$$

Ex. 73.

Write the second members of the following equations:

1. $(a^3)^2=$
2. $(x^5)^3=$
3. $(x^2y^3)^2=$
4. $\left(\frac{a^3b^2}{2}\right)^4=$
5. $\left(\frac{3x^2y}{2a^3b^2}\right)^5=$
6. $(2a^2bc^3)^4=$
7. $(-5ax^3y^2)^3=$
8. $(-7m^3nx^2y^4)^2=$
9. $\left(-\frac{2x^3y}{3abc}\right)^5=$
10. $(-3a^2b^2c)^5=$
11. $(-3xy^3)^6=$
12. $(-5a^2bx^3)^5=$
13. $\left(-\frac{3ab^2}{4c^3}\right)^4=$
14. $\left(-\frac{x^2y^3z^4}{2}\right)^7=$
15. $(x+2)^3=$
16. $(x-2)^4=$
17. $(x+3)^5=$
18. $(1+2x)^5=$
19. $(2m-1)^3=$
20. $(3x+1)^4=$
21. $(2x-a)^4=$
22. $(3x+2a)^5=$
23. $(2x-y)^4=$
24. $(x^2y-2xy^2)^6=$
25. $(ab-3)^7=$
26. $(1-a-a^2)^2=$
27. $(1-2x+x^2)^3=$
28. $(1-x+x^2)^3=$

EVOLUTION.

195. The operation of finding any required *root* of an expression is called **Evolution.**

Every case of evolution is merely an example of *factoring*, in which the required factors are all *equal*. Thus, the square, cube, fourth..... roots of an expression are found by taking one of the *two*, *three*, *four*..... *equal factors* of the expression.

196. The symbol which denotes that a square root is to be extracted is $\sqrt{\ }$; and for other roots the same symbol is used, but with a figure written above to indicate the root, thus, $\sqrt[3]{\ }$, $\sqrt[4]{\ }$, etc., signifies the *third* root, *fourth* root, etc.

197. Since the *cube* of $a^2 = a^6$, the *cube root* of $a^6 = a^2$.

Since the *fourth power* of $2a^2 = 2^4a^8$, the *fourth root* of $2^4a^8 = 2a^2$.

Since the square of $abc = a^2b^2c^2$, the *square root* of $a^2b^2c^2 = abc$.

Since the *square* of $\dfrac{ab}{xy} = \dfrac{a^2b^2}{x^2y^2}$, the *square root* of $\dfrac{a^2b^2}{x^2y^2} = \dfrac{ab}{xy}$.

Hence, the root of a simple expression is found by *dividing the exponent of each factor by the index of the root, and taking the product of the resulting factors.*

198. It is evident from § 192 that

I. Any *even* root of a *positive* number will have the double sign, $\pm$.

II. There can be no *even* root of a *negative* number.

III. Any *odd* root of a number will have the same sign as the number.

Thus, $\sqrt{\dfrac{16x^2}{81y^2}} = \pm\dfrac{4x}{9y}$; $\sqrt[3]{-27m^3n^6} = -3mn^2$;

$$\sqrt[4]{\frac{16x^8y^{12}}{81a^{16}}} = \pm\frac{2x^2y^3}{3a^4}.$$

But $\sqrt{-x^2}$ is neither $+x$ nor $-x$, for $(+x)^2 = +x^2$, and $(-x)^2 = +x^2$.

The indicated even root of a negative number is called an **impossible**, or **imaginary**, number.

199. If the root of a number expressed in figures is not readily detected, it may be found by resolving the number into its prime factors. Thus, to find the square root of 3,415,104:

$$\begin{array}{r|r} 2^3 & 3415104 \\ \hline 2^3 & 426888 \\ \hline 3^2 & 53361 \\ \hline 7 & 5929 \\ \hline 7 & 847 \\ \hline 11 & 121 \\ \hline & 11 \end{array}$$

$$\therefore \quad 3{,}415{,}104 = 2^6 \times 3^2 \times 7^2 \times 11^2.$$
$$\therefore \sqrt{3{,}415{,}104} = 2^3 \times 3 \times 7 \times 11 = 1848.$$

Ex. 74.

Simplify:

1. $\sqrt{a^4}, \sqrt[4]{x^8}, \sqrt{4a^6b^2}, \sqrt[3]{64}, \sqrt[5]{a^5x^{10}y^{15}}, \sqrt[4]{16a^{12}b^4c^8}, \sqrt[5]{-32a^{15}}.$

2. $\sqrt[3]{-1728c^6d^{12}x^3y^9}, \quad \sqrt[3]{3375b^{21}z^{15}}, \quad \sqrt[4]{3111696c^{16}z^4}.$

3. $\sqrt{53361b^4c^8y^{12}z^{16}}, \quad \sqrt[3]{-\dfrac{216b^3c^{15}}{343z^{24}}}, \quad \sqrt[6]{\dfrac{64x^{18}}{729z^{30}}}.$

4. $\sqrt{25a^2b^4c^2} + \sqrt[3]{8a^3b^6c^3} - \sqrt[4]{81a^4b^8c^4} - \sqrt[5]{32a^5b^{10}c^5}.$

5. $\sqrt[3]{27x^3y^6} \times \sqrt[5]{243y^5z^5} \times \sqrt{16x^4z^2}.$

When $a = 1$, $b = 3$, $x = 2$, $y = 6$, find the values of:

6. $4\sqrt{2x} - \sqrt{abxy} + 5\sqrt{a^2b^3xy}.$

7. $2a\sqrt{8ax} + b\sqrt[3]{12by} + 4abx\sqrt{bxy}.$

8. $\sqrt{a^2 + 2ab + b^2} \times \sqrt[3]{a^3 + 3a^2b + 3ab^2 + b^3}.$

9. $\sqrt[3]{b^3 - 3b^2a + 3ba^2 - a^3} \div \sqrt{b^2 + a^2 - 2ab}.$

SQUARE ROOTS OF COMPOUND EXPRESSIONS.

200. Since the square of $a+b$ is $a^2+2ab+b^2$, the square root of $a^2+2ab+b^2$ is $a+b$.

It is required to find a method of extracting the root $a+b$ when $a^2+2ab+b^2$ is given.

Ex. The first term, a, of the root is obviously the square root of the first term, a^2, in the expression.

$$\begin{array}{r|l}
 & a^2+2ab+b^2 \;\big|\; \underline{a+b} \\
 & \underline{a^2} \\
2a+b & 2ab+b^2 \\
 & \underline{2ab+b^2}
\end{array}$$

If the a^2 be subtracted from the given expression, the remainder is $2ab+b^2$. Therefore the second term, b, of the root is obtained when the first term of this remainder is divided by $2a$, that is, by *double the part of the root already found*. Also, since $2ab+b^2=(2a+b)b$, the divisor *is completed by adding to the trial-divisor the new term of the root.*

(1) Find the square root of $25x^2-20x^3y+4x^4y^2$.

$$\begin{array}{r|l}
 & 25x^2-20x^3y+4x^4y^2 \;\big|\; \underline{5x-2x^2y} \\
 & \underline{25x^2} \\
10x-2x^2y & -20x^3y+4x^4y^2 \\
 & \underline{-20x^3y+4x^4y^2}
\end{array}$$

The expression is *arranged* according to the ascending powers of x.

The square root of the first term is $5x$, and $5x$ is placed at the right of the given expression, for the first term of the root.

The second term of the root, $-2x^2y$, is obtained by dividing $-20x^3y$ by $10x$, and this new term of the root is also annexed to the divisor, $10x$, to complete the divisor.

201. The same method will apply to longer expressions, if care be taken to obtain the *trial-divisor* at each stage of the process, by *doubling the part of the root already found*, and to obtain the *complete divisor by annexing the new term of the root to the trial-divisor.*

Ex. Find the square root of

$$1 + 10x^2 + 25x^4 + 16x^6 - 24x^5 - 20x^3 - 4x.$$

$$\begin{array}{r|l}
 & 16x^6-24x^5+25x^4-20x^3+10x^2-4x+1 \;\underline{|\,4x^3-3x^2+2x-1} \\
 & 16x^6 \\
8x^3-3x^2 & -24x^5+25x^4 \\
 & \underline{-24x^5+\ 9x^4} \\
8x^3-6x^2+2x & 16x^4-20x^3+10x^2 \\
 & \underline{16x^4-12x^3+\ 4x^2} \\
8x^3-6x^2+4x-1 & -\ 8x^3+\ 6x^2-4x+1 \\
 & \underline{-\ 8x^3+\ 6x^2-4x+1}
\end{array}$$

The expression is arranged according to the descending powers of x.

It will be noticed that each successive trial-divisor may be obtained by taking the preceding complete divisor with its *last term doubled.*

Ex. 75.

Extract the square roots of:

1. $a^4 + 4a^3 + 2a^2 - 4a + 1.$
2. $x^4 - 2x^3y + 3x^2y^2 - 2xy^3 + y^4.$
3. $4a^6 - 12a^5x + 5a^4x^2 + 6a^3x^3 + a^2x^4.$
4. $9x^6 - 12x^3y^3 + 16x^2y^4 - 24x^4y^2 + 4y^6 + 16xy^5.$
5. $4a^8 + 16c^8 + 16a^6c^2 - 32a^2c^6.$
6. $4x^4 + 9 - 30x - 20x^3 + 37x^2.$
7. $16x^4 - 16abx^2 + 16b^2x^2 + 4a^2b^2 - 8ab^3 + 4b^4.$
8. $x^6 + 25x^2 + 10x^4 - 4x^5 - 20x^3 + 16 - 24x.$
9. $x^6 + 8x^4y^2 - 4x^5y - 4xy^5 + 8x^2y^4 - 10x^3y^3 + y^6.$
10. $4 - 12a - 11a^4 + 5a^2 - 4a^5 + 4a^6 + 14a^3.$
11. $25x^6 - 31x^4y^2 + 34x^3y^3 - 30x^5y + y^6 - 8xy^5 + 10x^2y^4.$

12. $x^4 - x^3y - \frac{7}{4}x^2y^2 + xy^3 + y^4.$

13. $x^4 - 4x^3y + 6x^2y^2 - 6xy^3 + 5y^4 - \frac{2y^5}{x} + \frac{y^6}{x^2}.$

Square Roots of Arithmetical Numbers.

202. In the general method of extracting the square root of a number expressed by figures, the first step is to mark off the figures in *periods*.

Since $1 = 1^2$, $100 = 10^2$, $10{,}000 = 100^2$, and so on, it is evident that the square root of any number between 1 and 100 lies between 1 and 10; the square root of any number between 100 and 10,000 lies between 10 and 100. In other words, the square root of any number expressed by *one* or *two* figures is a number of *one* figure; the square root of any number expressed by *three* or *four* figures is a number of *two* figures; and so on.

If, therefore, a dot be placed over the *units' figure* of a square number, and over every *alternate* figure, the number of dots will be equal to the number of figures in its square root.

Find the square root of 3249.

$$\begin{array}{r|l} & 3\dot{2}4\dot{9}\,(57 \\ & \underline{25} \\ 107) & 749 \\ & \underline{749} \end{array}$$

In this case, a in the typical form $a^2 + 2ab + b^2$ represents 5 *tens*, that is, 50, and b represents 7. The 25 subtracted is really 2500, that is, a^2, and the complete divisor, $2a + b$, is $2 \times 50 + 7 = 107$.

203. The same method will apply to numbers of more than two periods by considering a in the typical form to represent at each step *the part of the root already found.*

It must be observed that *a represents so many tens with respect to the next figure of the root.*

Ex. Find the square root of 5,322,249.

$$\begin{array}{r|l} & \dot{5}3\dot{2}2\dot{2}4\dot{9}\,(2307 \\ & \underline{4} \\ 43) & 132 \\ & \underline{129} \\ 4607) & 32249 \\ & \underline{32249} \end{array}$$

204. If the square root of a number have decimal places, the number itself will have *twice* as many.

Thus, if .21 be the square root of some number, this number will be $(.21)^2 = .21 \times .21 = .0441$; and if .111 be the root, the number will be $(.111)^2 = .111 \times .111 = .012321$.

Therefore, the number of *decimal* places in every square decimal will be *even*, and the number of decimal places in the root will be *half* as many as in the given number itself.

Hence, if the given square number contain a decimal, and a dot be placed over the *units' figure*, and then over every *alternate* figure on *both* sides of it, the number of dots to the left of the decimal point will show the number of *integral* places in the root, and the number of dots to the right will show the number of *decimal* places.

Ex. Find the square roots of 41.2164 and 965.9664.

$$\begin{array}{rl} & 4\dot{1}.2\dot{1}6\dot{4}\,(6.42 \\ & \underline{36} \\ 124) & 521 \\ & \underline{496} \\ 1282) & 2564 \\ & \underline{2564} \end{array} \qquad \begin{array}{rl} & \dot{9}6\dot{5}.9\dot{6}6\dot{4}\,(31.08 \\ & \underline{9} \\ 61) & 65 \\ & \underline{61} \\ 6208) & 49664 \\ & \underline{49664} \end{array}$$

It is seen from the dotting that the root of the first example will have one integral and two decimal places, and that the root of the second example will have two integral and two decimal places.

205. If a number contain an *odd* number of decimal places, or if any number give a *remainder* when as many figures in the root have been obtained as the given number has periods, then its exact square root cannot be found. We may, however, approximate to its exact root as near as we please by annexing ciphers and continuing the operation.

Ex. Find the square roots of 3 and 357.357.

$$\begin{array}{rl}
 & \dot{3}.(1.732\ldots\ldots \\
 & 1 \\
27) & 200 \\
 & 189 \\
343) & 1100 \\
 & 1029 \\
3462) & 7100 \\
 & 6924
\end{array}
\qquad\qquad
\begin{array}{rl}
 & \dot{3}5\dot{7}.3\dot{5}7\dot{0}(18.903\ldots\ldots \\
 & 1 \\
28) & 257 \\
 & 224 \\
369) & 3335 \\
 & 3321 \\
37803) & 147000 \\
 & 113409
\end{array}$$

Ex. 76.

Extract the square roots of:

1. 120,409; 4816.36; 1867.1041; 1435.6521; 64.128064.
2. 16,803.9369; 4.54499761; .24373969; .5687573056.
3. .9; 6.21; .43; .00852; 17; 129; 347.259.
4. 14,295.387; 2.5; 2000; .3; .03; 111.
5. .00111; .004; .005; 2; 5; 3.25; 8.6.
6. $\frac{1}{4}$; $\frac{16}{49}$; $\frac{100}{144}$; $\frac{169}{225}$; $\frac{289}{324}$; $\frac{400}{625}$.
7. $\frac{1}{2}$; $\frac{2}{3}$; $\frac{3}{4}$; $\frac{1}{32}$; $\frac{7}{128}$; $\frac{6}{125}$; $\frac{6}{7}$; $\frac{1}{12}$.

Cube Roots of Compound Expressions.

206. Since the cube of $a+b$ is $a^3+3a^2b+3ab^2+b^3$, the cube root of $a^3+3a^2b+3ab^2+b^3$ is $a+b$.

It is required to find a method for extracting the cube root $a+b$ when $a^3+3a^2b+3ab^2+b^3$ is given:

(1) Find the cube root of $a^3+3a^2b+3ab^2+b^3$.

$$\begin{array}{r|l}
 & a^3+3a^2b+3ab^2+b^3 \,\underline{|\, a+b} \\
3a^2 \qquad\qquad & a^3 \\
+3ab+b^2 & 3a^2b+3ab^2+b^3 \\
\hline
3a^2+3ab+b^2 & 3a^2b+3ab^2+b^3
\end{array}$$

The first term a of the root is obviously the cube root of the first term a^3 of the given expression.

If a^3 be subtracted, the remainder is $3a^2b + 3ab^2 + b^3$; therefore, the second term b of the root is obtained by dividing the first term of this remainder by *three times the square of* a.

Also, since $3a^2b + 3ab^2 + b^3 = (3a^2 + 3ab + b^2)b$, the *complete divisor* is obtained by adding $3ab + b^2$ to the *trial-divisor* $3a^2$.

(2) Find the cube root of $8x^3+36x^2y+54xy^2+27y^3$.

$$
\begin{array}{rr|l}
 & 8x^3+36x^2y+54xy^2+27y^3 & \underline{2x+3y} \\
12x^2 & 8x^3 & \\
(6x+3y)3y = \quad 18xy+9y^2 & 36x^2y+54xy^2+27y^3 & \\
\hline
12x^2+18xy+9y^2 & 36x^2y+54xy^2+27y^3 &
\end{array}
$$

The cube root of the first term is $2x$, and this is therefore the first term of the root.

The second term of the root, $3y$, is obtained by dividing $36x^2y$ by $3(2x)^2 = 12x^2$, which corresponds to $3a^2$ in the typical form, and is completed by annexing to $12x^2$ the expression $\{3(2x)+3y\}3y = 18xy + 9y^2$, which corresponds to $3ab + b^2$, in the typical form.

207. The same method may be applied to longer expressions by considering a in the typical form $3a^2 + 3ab + b^2$ to represent at each stage of the process *the part of the root already found.*

Thus, if the part of the root already found be $x + y$, then $3a^2$ of the typical form will be represented by $3(x+y)^2$; and if the third term of the root be $+z$, the $3ab + b^2$ will be represented by $3(x+y)z + z^2$. So that the complete divisor, $3a^2 + 3ab + b^2$, will be represented by $3(x+y)^2 + 3(x+y)z + z^2$.

Find the cube root of $x^6 - 3x^5 + 5x^3 - 3x - 1$.

$$
\begin{array}{rr|l}
 & & \underline{x^2 - x - 1} \\
 & & x^6 - 3x^5 + 5x^3 - 3x - 1 \\
 & 3x^4 & x^6 \\
(3x^2 - x)(-x) = & -3x^3 + x^2 & -3x^5 + 5x^3 \\
 & 3x^4 - 3x^3 + x^2 & -3x^5 + 3x^4 - x^3 \\
\hline
 & & -3x^4 + 6x^3 - 3x - 1 \\
3(x^2 - x)^2 = & 3x^4 - 6x^3 + 3x^2 & \\
(3x^2 - 3x - 1)(-1) = & -3x^2 + 3x + 1 & \\
\hline
 & 3x^4 - 6x^3 \quad + 3x + 1 & -3x^4 + 6x^3 - 3x - 1
\end{array}
$$

The root is placed above the given expression for convenience of arrangement.

The first term of the root, x^2, is obtained by taking the cube root of the first term of the given expression; and the first trial-divisor, $3x^4$, is obtained by taking three times the square of this term of the root.

The first complete divisor is found by annexing to the trial-divisor $(3x^2 - x)(-x)$, which expression corresponds to $(3a + b)b$ in the typical form.

The part of the root already found (a) is now represented by $x^2 - x$; therefore $3a^2$ is represented by $3(x^2 - x)^2 = 3x^4 - 6x^3 + 3x^2$, the second trial-divisor; and $(3a + b)b$ by $(3x^2 - 3x - 1)(-1)$; therefore, in the second complete divisor, $3a^2 + (3a + b)b$ is represented by

$$(3x^4 - 6x^3 + 3x^2) + (-3x^2 - 3x - 1) \times (-1) = 3x^4 - 6x^3 + 3x + 1.$$

Ex. 77.

Find the cube roots of:

1. $x^3 + 6x^2y + 12xy^2 + 8y^3$.
2. $a^3 - 9a^2 + 27a - 27$.
3. $x^3 + 12x^2 + 48x + 64$.
4. $x^6 - 3ax^5 + 5a^3x^3 - 3a^5x - a^6$.
5. $x^6 + 3x^5 + 6x^4 + 7x^3 + 6x^2 + 3x + 1$.
6. $1 - 9x + 39x^2 - 99x^3 + 156x^4 - 144x^5 + 64x^6$.
7. $a^6 - 6a^5 + 9a^4 + 4a^3 - 9a^2 - 6a - 1$.
8. $64x^6 + 192x^5 + 144x^4 - 32x^3 - 36x^2 + 12x - 1$.
9. $1 - 3x + 6x^2 - 10x^3 + 12x^4 - 12x^5 + 10x^6 - 6x^7 + 3x^8 - x^9$.
10. $a^6 + 9a^5b - 135a^3b^3 + 729ab^5 - 729b^6$.
11. $c^6 - 12bc^5 + 60b^2c^4 - 160b^3c^3 + 240b^4c^2 - 192b^5c + 64b^6$.
12. $8x^6 + 48a^5b + 60a^4b^2 - 80a^3b^3 - 90a^2b^4 + 108ab^5 - 27b^6$.

Cube Roots of Arithmetical Numbers.

208. In extracting the cube root of a number expressed by figures, the first step is to mark it off into periods.

Since $1 = 1^3$, $1000 = 10^3$, $1,000,000 = 100^3$, and so on, it follows that the cube root of any number between 1 and 1000, that is, of any number which has *one, two*, or *three* figures, is a number of *one* figure; and that the cube root of any number between 1000 and 1,000,000, that is, of any number which has *four, five*, or *six* figures, is a number of *two* figures; and so on.

Hence, if a dot be placed over every *third* figure of a cube number, beginning with the *units' figure*, the number of dots will be equal to the number of figures in its cube root.

209. If the cube root of a number contain any decimal figures, the number itself will contain *three times* as many.

Thus, if .3 be the cube root of a number, the number is $.3 \times .3 \times .3 = .027$.

Hence, if the given cube number have decimal places, and a dot be placed over the *units' figure* and over every *third* figure on *both* sides of it, the number of dots to the *left* of the decimal point will show the number of *integral* figures in the root; and the number of dots to the *right* will show the number of *decimal* figures in the root.

If the given number be not a perfect cube, ciphers may be annexed, and a value of the root may be found as near to the *true* value as we please.

210. It is to be observed that if a denote the first term of the root, and b the second term, the *first complete divisor* is

$$3a^2 + 3ab + b^2,$$

and the *second trial-divisor* is $3(a+b)^2$, that is,

$$3a^2 + 6ab + 3b^2,$$

which may be obtained from the preceding complete divisor by adding to it its second term and twice its third term,

$$\begin{array}{r} 3a^2 + 3ab + b^2 \\ + 3ab + 2b^2 \\ \hline 3a^2 + 6ab + 3b^2 \end{array}$$

a method which will very much shorten the work in long arithmetical examples.

211. Ex. Extract the cube root of 5 to five places of decimals.

$$\begin{array}{rr|l}
 & & 5.000(1.70997 \\
 & & 1 \\
3\times 10^2 = & 300 & 4\,000 \\
3(10\times 7) = & 210 & \\
7^2 = & 49 & \\
 & 559 & 3\,913 \\
 & 259 & 87\,000\,000 \\
3\times 1700^2 = & 8670000 & \\
3(1700\times 9) = & 45900 & \\
9^2 = & 81 & \\
 & 8715981 & 78\,443\,829 \\
 & 45981 & 8\,556\,1710 \\
3\times 1709^2 = & 8762043 & 7\,885\,8387 \\
 & & 670\,33230 \\
 & & 613\,34301
\end{array}$$

After the first two figures of the root are found, the next trial divisor is obtained by bringing down the sum of the 210 and 49 obtained in completing the preceding divisor; then adding the three lines connected by the brace, and annexing two ciphers to the result.

The last two figures of the root are found by division. The rule in such cases is, that two less than the number of figures already obtained may be found without error by division, the divisor to be employed being three times the square of the part of the root already found.

Ex. 78.

Find the cube roots of:

1. 274,625.
2. 110,592.
3. 262,144.
4. 884.736.
5. 109,215,352.
6. 1,481,544.
7. 1601.613.
8. 1,259.712.
9. 2.803221.
10. 7,077,888.
11. 12.812904.
12. 56.623104.

13. 33,076.161. **15.** 820.025856. **17.** 1.371330631.

14. 102,503.232. **16.** 8653.002877. **18.** 20,910.518875.

19. 91.398648466125. **20.** 5.340104393239.

21. Find to four figures the cube roots of 2.5; .2; .01; 4; .4.

212. Since the fourth power is the square of the square, and the sixth power the square of the cube, the *fourth root* is the *square root* of the *square root*, and the *sixth root* is the *cube root* of the *square root*. In like manner, the eighth, ninth, twelfth..... roots may be found.

Ex. 79.

Find the fourth roots of:

1. $81a^4 - 540a^3b + 1350a^2b^2 - 1500ab^3 + 625b^4$.

2. $1 - 4x + 10x^2 - 16x^3 + 19x^4 - 16x^5 + 10x^6 - 4x^7 + x^8$.

Find the sixth roots of:

3. $64 - 192x + 240x^2 - 160x^3 + 60x^4 - 12x^5 + x^6$.

4. $729x^6 - 1458x^5 + 1215x^4 - 540x^3 + 135x^2 - 18x + 1$.

Find the eighth root of:

5. $1 - 8y + 28y^2 - 56y^3 + 70y^4 - 56y^5 + 28y^6 - 8y^7 + y^8$.

CHAPTER XIV.

QUADRATIC EQUATIONS.

213. An equation which contains the *square* of the unknown quantity, but no higher power, is called a **quadratic equation.**

214. If the equation contain the *square only*, it is called a **pure quadratic**; but if it contain the *first power also*, it is called an **affected quadratic.**

PURE QUADRATIC EQUATIONS.

Solve the equation $5x^2 - 48 = 2x^2$.

$$5x^2 - 48 = 2x^2$$
$$3x^2 = 48$$
$$x^2 = 16$$
$$\therefore x = \pm 4$$

It will be observed that there are *two* roots of equal value but of opposite signs; and there are only two, for if the square root of the equation $x^2 = 16$ were written $\pm x = \pm 4$, there would be only two values of x; since the equation $-x = +4$ gives $x = -4$, and the equation $-x = -4$ gives $x = 4$.

Hence, to solve a pure quadratic,

Collect the unknown quantities on one side, and the known quantities on the other; divide by the coefficient of the unknown quantity; and extract the square root of each side of the resulting equation.

Solve the equation $3x^2 - 15 = 0$.

$$3x^2 - 15 = 0$$
$$3x^2 = 15$$
$$x^2 = 5$$
$$\therefore x = \pm\sqrt{5}$$

It will be observed that the square root of 5 cannot be found exactly, but an approximate value of it to any assigned degree of accuracy may be found.

215. A root which is indicated, but which can be found only approximately, is called a **Surd.**

Solve the equation $3x^2 + 15 = 0$.

$$3x^2 + 15 = 0$$
$$3x^2 = -15$$
$$x^2 = -5$$
$$\therefore x = \pm\sqrt{-5}$$

It will be observed that the square root of -5 cannot be found even approximately; for the square of any number, positive or negative, is positive.

216. A root which is indicated, but which cannot be found exactly or approximately, is **imaginary.** § 198.

Ex. 80.

Solve:

1. $x^2 - 3 = 46.$
2. $2(x^2-1)-3(x^2+1)+14=0.$
3. $\dfrac{x^2-5}{3} + \dfrac{2x^2+1}{6} = \dfrac{1}{2}.$
4. $\dfrac{3}{1+x} + \dfrac{3}{1-x} = 8.$
5. $\dfrac{3}{4x^2} - \dfrac{1}{6x^2} = \dfrac{7}{3}.$
6. $5x^2 - 9 = 2x^2 + 24.$
7. $(x+2)^2 = 4x + 5.$
8. $\dfrac{x^2}{5} - \dfrac{x^2-10}{15} = 7 - \dfrac{50+x^2}{25}.$
9. $\dfrac{3x^2-27}{x^2+3} + \dfrac{90+4x^2}{x^2+9} = 7.$
10. $8x + \dfrac{7}{x} = \dfrac{65x}{7}.$
11. $\dfrac{4x^2+5}{10} - \dfrac{2x^2-5}{15} = \dfrac{7x^2-25}{20}.$
12. $\dfrac{10x^2+17}{18} - \dfrac{12x^2+2}{11x^2-8} = \dfrac{5x^2-4}{9}.$
13. $\dfrac{14x^2+16}{21} - \dfrac{2x^2+8}{8x^2-11} = \dfrac{2x^2}{3}.$
14. $x^2 + bx + a = bx(1-bx).$
15. $mx^2 + n = q.$
16. $x^2 - ax + b = ax(x-1).$

AFFECTED QUADRATIC EQUATIONS.

217. Since $(ax \pm b)^2 = a^2x^2 \pm 2abx + b^2$, it is evident that the expression $a^2x^2 \pm 2abx$ lacks only the *third term*, b^2, of being a complete square.

It will be seen that this third term is *the square of the quotient obtained from dividing the second term by twice the square root of the first term.*

218. Every affected quadratic may be made to assume the form of $a^2x^2 \pm 2abx = c$.

The first step in the solution of such an equation is to *complete the square;* that is, to add to each side *the square of the quotient obtained from dividing the second term by twice the square root of the first term.*

The second step is to *extract the square root* of each side of the resulting equation.

The third and last step is to *reduce* the resulting simple equation.

(1) Solve the equation $16x^2 + 5x - 3 = 7x^2 - x + 45$.

$$16x^2 + 5x - 3 = 7x^2 - x + 45.$$

Simplify, $9x^2 + 6x = 48.$

Complete the square, $9x^2 + 6x + 1 = 49.$

Extract the root, $3x + 1 = \pm 7.$

Reduce, $3x = -1 + 7 \text{ or } -1 - 7,$

$$3x = 6 \text{ or } -8.$$

$$\therefore\ x = 2 \text{ or } -2\tfrac{2}{3}.$$

Verify by substituting 2 for x in the equation,

$$16x^2 + 5x - 3 = 7x^2 - x + 45$$
$$16(2)^2 + 5(2) - 3 = 7(2)^2 - (2) + 45$$
$$64 + 10 - 3 = 28 - 2 + 45$$
$$71 = 71.$$

Verify by substituting $-2\frac{2}{3}$ for x in the equation,

$$
\begin{aligned}
16x^2 + 5x - 3 &= 7x^2 - x + 45\\
16(-\tfrac{8}{3})^2 + 5(-\tfrac{8}{3}) - 3 &= 7(-\tfrac{8}{3})^2 - (-\tfrac{8}{3}) + 45\\
\tfrac{1024}{9} - \tfrac{40}{3} - 3 &= \tfrac{448}{9} + \tfrac{8}{3} + 45\\
1024 - 120 - 27 &= 448 + 24 + 405\\
877 &= 877.
\end{aligned}
$$

(2) Solve the equation $3x^2 - 4x = 32$.

Since the exact root of 3, the coefficient of x^2, cannot be found, it is necessary to multiply or divide each term of the equation by 3 to make the coefficient of x^2 a *square number*.

Multiply by 3,	$9x^2 - 12x = 96.$
Complete the square,	$9x^2 - 12x + 4 = 100.$
Extract the root,	$3x - 2 = \pm 10.$
Reduce,	$3x = 2 + 10 \text{ or } 2 - 10$
	$3x = 12 \text{ or } -8.$
	$\therefore x = 4 \text{ or } -2\frac{2}{3}.$

Or, divide by 3,	$x^2 - \frac{4x}{3} = \frac{32}{3}.$
Complete the square,	$x^2 - \frac{4x}{3} + \frac{4}{9} = \frac{32}{3} + \frac{4}{9} = \frac{100}{9}.$
Extract the root,	$x - \frac{2}{3} = \pm \frac{10}{3}.$
	$\therefore x = \frac{2 \pm 10}{3}$
	$= 4 \text{ or } -2\frac{2}{3}.$

Verify by substituting 4 for x in the original equation,

$$
\begin{aligned}
48 - 16 &= 32\\
32 &= 32.
\end{aligned}
$$

Verify by substituting $-2\frac{2}{3}$ for x in the original equation,

$$
\begin{aligned}
21\tfrac{1}{3} - (-10\tfrac{2}{3}) &= 32\\
32 &= 32.
\end{aligned}
$$

(3) Solve the equation $-3x^2+5x=-2$.

Since the *even* root of a *negative* number is impossible, it is necessary to change the sign of each term. The resulting equation is

$$3x^2-5x=2.$$

Multiply by 3, $\quad 9x^2-15x=6.$

Complete the square, $\quad 9x^2-15x+\frac{25}{4}=\frac{49}{4}.$

Extract the root, $\quad 3x-\frac{5}{2}=\pm\frac{7}{2}.$

Reduce, $\quad 3x=\frac{5\pm 7}{2}$

$$3x=6 \text{ or } -1.$$
$$\therefore x=2 \text{ or } -\tfrac{1}{3}.$$

Or, divide by 3, $\quad x^2-\frac{5x}{3}=\frac{2}{3}.$

Complete the square, $\quad x^2-\frac{5x}{3}+\frac{25}{36}=\frac{49}{36}.$

Extract the root, $\quad x-\frac{5}{6}=\pm\frac{7}{6}.$

$$\therefore x=\frac{5\pm 7}{6}$$
$$=2 \text{ or } -\tfrac{1}{3}.$$

If the equation $3x^2-5x=2$ be multiplied by *four times the coefficient of* x^2, fractions will be avoided:

$$36x^2-60x=24.$$

Complete the square, $\quad 36x^2-60x+25=49.$

Extract the root, $\quad 6x-5=\pm 7$

$$6x=5\pm 7$$
$$6x=12 \text{ or } -2.$$
$$\therefore x=2 \text{ or } -\tfrac{1}{3}.$$

It will be observed that the number added to complete the square by this last method is *the square of the coefficient of* x in the original equation $3x^2-5x=2$.

(4) Solve the equation $\dfrac{3}{5-x} - \dfrac{1}{2x-5} = 2.$

Simplify (as in simple equations),

$$4x^2 - 23x = -30.$$

Multiply by four times the coefficient of x^2, and add to each side the square of the coefficient of x,

$$64x^2 - (\ ) + (23)^2 = 529 - 480 = 49.$$

Extract the root, $\quad 8x - 23 = \pm 7.$

Reduce, $\quad 8x = 23 \pm 7$

$$8x = 30 \text{ or } 16.$$

$$\therefore x = 3\tfrac{3}{4} \text{ or } 2.$$

If a trinomial be a perfect square, its root is found by taking the roots of the *first* and *third* terms and connecting them by the *sign* of the middle term. It is not necessary, therefore, in completing the square, to write the middle term, but its place may be indicated as in this example.

(5) Solve the equation $72x^2 - 30x = -7.$

Since $72 = 2^3 \times 3^2$, if the equation be multiplied by 2, the coefficient of x^2 in the resulting equation, $144x^2 - 60x = -14$, will be a square number, and the term required to complete the square will be $(\frac{60}{24})^2 = (\frac{5}{2})^2 = \frac{25}{4}$. Hence, if the original equation be multiplied by 4×2, the coefficient of x^2 in the result will be a square number, and fractions will be avoided in the work.

Multiply the given equation by 8,

$$576x^2 - 240x = -56.$$

Complete the square, $\quad 576x^2 - (\ ) + 25 = -31.$

Extract the root, $\quad 24x - 5 = \pm\sqrt{-31}.$

Reduce, $\quad 24x = 5 \pm \sqrt{-31}.$

$$\therefore x = \tfrac{1}{24}(5 \pm \sqrt{-31}).$$

NOTE. In solving the following equations, care must be taken to select the method best adapted to the example under consideration.

Ex. 81.

Solve:

1. $x^2 + 4x = 12.$
2. $x^2 - 6x = 16.$
3. $x^2 - 12x + 6 = \frac{1}{4}.$
4. $x^2 - 7x = 8.$
5. $3x^2 - 4x = 7.$
6. $12x^2 + x - 1 = 0.$
7. $x^2 - x = 6.$
8. $5x^2 - 3x = 2.$
9. $2x^2 - 27x = 14.$

10. $x^2 - \frac{2x}{3} + \frac{1}{12} = 0.$

11. $\frac{x^2}{2} - \frac{x}{3} = 2(x+2).$

12. $\frac{3x}{4} + \frac{4}{3x} = \frac{13}{6}.$

13. $\frac{x+1}{x+4} = \frac{2x-1}{x+6}.$

14. $\frac{x}{x+1} - \frac{x+3}{2(x+4)} = -\frac{1}{18}.$

15. $\frac{2}{x-1} = \frac{3}{x-2} + \frac{2}{x-4}.$

16. $5x(x-3) - 2(x^2-6) = (x+3)(x+4).$

17. $\frac{3x}{2(x+1)} - \frac{5}{8} = \frac{3x^2}{x^2-1} - \frac{23}{4(x-1)}.$

18. $(x-2)(x-4) - 2(x-1)(x-3) = 0.$

19. $\frac{1}{7}(x-4) - \frac{2}{5}(x-2) = \frac{1}{x}(2x+3).$

20. $\frac{2}{5}(3x^2 - x - 5) - \frac{1}{3}(x^2-1) = 2(x-2)^2.$

21. $\frac{x+2}{x-1} - \frac{4-x}{2x} = \frac{7}{3}.$

22. $1 - \frac{x+5}{2x+1} = \frac{x-6}{x-2}.$

219. *Literal quadratic equations* are solved as follows:

(1) Solve the equation $ax^2 + bx = c.$

Multiply the equation by $4a$ and add the square of b,

$$4a^2x^2 + (\quad) + b^2 = 4ac + b^2.$$

Extract the root, $$2ax + b = \pm\sqrt{4ac + b^2}.$$

Reduce, $$2ax = -b \pm \sqrt{4ac + b^2}.$$

$$\therefore x = \frac{-b \pm \sqrt{4ac + b^2}}{2a}$$

(2) Solve the equation $adx - acx^2 = bcx - bd.$

Transpose bcx and change the signs,

$$acx^2 + bcx - adx = bd.$$

Express the left member in *two terms*,

$$acx^2 + (bc - ad)x = bd.$$

Multiply by $4ac$,

$$4a^2c^2x^2 + 4ac(bc - ad)x = 4abcd.$$

Complete the square,

$$4a^2c^2x^2 + (\quad) + (bc - ad)^2 = b^2c^2 + 2abcd + a^2d^2.$$

Extract the root, $2acx + (bc - ad) = \pm(bc + ad).$

Reduce,

$$2acx = -(bc - ad) \pm (bc + ad)$$
$$= 2ad, \text{ or } -2bc.$$
$$\therefore x = \frac{d}{c}, \text{ or } -\frac{b}{a}.$$

(3) Solve the equation $px^2 - px + qx^2 + qx = \dfrac{pq}{p+q}.$

Express the left member in *two terms*,

$$(p+q)x^2 - (p-q)x = \frac{pq}{p+q}.$$

Multiply by four times the coefficient of x^2,

$$4(p+q)^2x^2 - 4(p^2 - q^2)x = 4pq.$$

Complete the square,

$$4(p+q)^2x^2 - (\quad) + (p-q)^2 = p^2 + 2pq + q^2.$$

Extract the root, $2(p+q)x - (p-q) = \pm(p+q).$

Reduce,

$$2(p+q)x = (p-q) \pm (p+q)$$
$$= 2p, \text{ or } -2q.$$
$$\therefore x = \frac{p}{p+q}, \text{ or } -\frac{q}{p+q}.$$

NOTE. The left-hand member of the equation when simplified must be expressed in *two terms, simple or compound*, one term containing x^2, and the other term containing x.

Ex. 82.

Solve:

1. $x^2 + 2ax = a^2.$
2. $x^2 = 4ax + 7a^2.$
3. $x^2 = \dfrac{7m^2}{4} - 3mx.$
4. $x^2 - \dfrac{5nx}{2} - \dfrac{3n^2}{2} = 0.$
5. $\dfrac{a^2}{(x+a)^2} = \dfrac{b^2}{(x-a)^2}.$
6. $cx = ax^2 + bx^2 - \dfrac{ac}{a+b}.$
7. $\dfrac{a^2x^2}{b^2} + \dfrac{b^2}{c^2} = \dfrac{2ax}{c}.$
8. $(a^2+1)x = ax^2 + a.$
9. $\dfrac{a}{x-a} + \dfrac{b}{x-b} = \dfrac{2c}{x-c}.$
10. $\dfrac{1}{a+b+x} = \dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{x}.$

11. $\dfrac{1}{a-x}-\dfrac{1}{a+x}=\dfrac{3+x^2}{a^2-x^2}.$

12. $\dfrac{(2x-a)^2}{2x-a+2b}=b.$

13. $x^2+ax=a+x.$

14. $x^2+ax=bx+ab.$

15. $\dfrac{x}{a}+\dfrac{a}{x}=\dfrac{x}{b}+\dfrac{b}{x}.$

16. $\dfrac{1}{x}+\dfrac{1}{x+b}=\dfrac{1}{a}+\dfrac{1}{a+b}.$

17. $\dfrac{a}{3}+\dfrac{5x}{4}-\dfrac{x^2}{3a}=0.$

18. $\dfrac{x+3}{x-3}=a+\dfrac{x-3}{x+3}.$

19. $(ax-b)(bx-a)=c^2.$

20. $abx^2+\dfrac{b^2x}{c}=\dfrac{6a^2+ab-2b^2}{c^2}-\dfrac{3a^2x}{c}.$

21. $\dfrac{x^2}{3m-2a}-\dfrac{m^2-4a^2}{4a-6m}=\dfrac{x}{2}.$

22. $6x+\dfrac{(a+b)^2}{x}=5(a-b)+\dfrac{25ab}{6x}.$

23. $\dfrac{x+13a+3b}{5a-3b-x}-1=\dfrac{a-2b}{x+2b}.$

220. An affected quadratic may be reduced to the form $x^2+px+q=0$, in which p and q represent *any* numbers, positive or negative, integral or fractional.

Ex. Solve $\quad x^2+px+q=0.$

$$4x^2+(\ )+p^2=p^2-4q$$
$$2x+p=\pm\sqrt{p^2-4q}.$$
$$\therefore\ x=-\frac{p}{2}\pm\frac{1}{2}\sqrt{p^2-4q}.$$

By this formula, the values of x in an equation of the form $x^2+px+q=0$, may be written at once. Thus, take the equation

$$3x^2-5x+2=0.$$

Divide by 3, $x^2 - \frac{5}{3}x + \frac{2}{3} = 0.$

Here, $p = -\frac{5}{3}$, and $q = \frac{2}{3}$.

$$\therefore x = \frac{5}{6} \pm \frac{1}{2}\sqrt{\frac{25}{9} - \frac{8}{3}}$$
$$= \frac{5}{6} \pm \frac{1}{6}$$
$$= 1, \text{ or } \frac{2}{3}.$$

221. A quadratic which has been reduced to its simplest form, and has all its terms written on one side, may often have that side resolved *by inspection* into factors.

In this case, the roots are seen at once without completing the square.

(1) Solve $x^2 + 7x - 60 = 0$.

Since $x^2 + 7x - 60 = (x + 12)(x - 5)$,
the equation $x^2 + 7x - 60 = 0$
may be written $(x + 12)(x - 5) = 0$.

It will be observed that if *either* of the factors $x + 12$ or $x - 5$ is 0, the *product of the two factors* is 0, and the equation is satisfied.

Hence, $x + 12 = 0$ and $x - 5 = 0$.

$\therefore x = -12$, and $x = 5$.

(2) Solve $x^2 + 7x = 0$.

The equation $x^2 + 7x = 0$
becomes $x(x + 7) = 0$,
and is satisfied if $x = 0$, or if $x + 7 = 0$.

$\therefore$ the roots are 0 and -7.

It will be observed that this method is easily applied to an equation *all* the terms of which contain x.

(3) Solve $2x^3 - x^2 - 6x = 0$.

The equation $2x^3 - x^2 - 6x = 0$
becomes $x(2x^2 - x - 6) = 0$,
and is satisfied if $x = 0$, or if $2x^2 - x - 6 = 0$.

By solving $2x^2 - x - 6 = 0$, the two roots 2 and $-\frac{3}{2}$ are found.

$\therefore$ the equation has *three* roots, 0, 2, $-\frac{3}{2}$.

(4) Solve $x^3+x^2-4x-4=0$.

The equation $\quad x^3+x^2-4x-4=0$

becomes $\quad x^2(x+1)-4(x+1)=0$

$$(x^2-4)(x+1)=0.$$

$\therefore$ the roots of the equation are $-1, 2, -2$.

(5) Solve $x^3-2x^2-11x+12=0$.

Since $$\frac{x^3-2x^2-11x+12}{x-1}=x^2-x-12,$$

the equation $\quad x^3-2x^2-11x+12=0$

may be written $\quad (x-1)(x^2-x-12)=0.$

The three roots are found to be $1, -3, 4$.

An equation which cannot be resolved into factors by inspection may sometimes be solved by *guessing* at a root, and reducing by division. In this case, if a denote the root, the given equation (all the terms of the equation being written on one side) may be divided by $x-a$.

Ex. 83.

Find the roots of:

1. $(x+1)(x-2)(x^2+x-2)=0$.
2. $(x^2-3x+2)(x^2-x-12)=0$.
3. $(x+1)(x-2)(x+3)=-6$.
4. $2x^3+4x^2-70x=0$.
5. $(x^2-x-6)(x^2-x-20)=0$.
6. $x^3-x^2-x+1=0$.
7. $8x^3-1=0$.
8. $8x^3+1=0$.

222. If r and r' represent two values of x, then

$$x-r=0,$$

and $$x-r'=0.$$

$$\therefore (x-r)(x-r')=0.$$

This is a quadratic equation, as may be seen by performing the indicated multiplication.

Now r and r' are roots of this equation; for, if either r or r' be written for x, one of the factors, $x-r$, $x-r'$, is equal to 0, and the equation is satisfied. Also r and r' are the *only* roots, for no value of x, except r and r', can make either of these factors equal to 0.

Since r and r' may represent the values of x in any quadratic equation, it follows that every quadratic equation has *two* roots, and *only two*.

Again, if r, r', r'', represent three values of x,

then $$(x-r)(x-r')(x-r'')=0.$$

This is a *cubic* equation, as may be seen by performing the indicated multiplication. Hence, it may be inferred that a *cubic* equation has *three* roots, and *only three*; and so, for any equation, that the *number* of roots is equal to the *degree* of the equation.

It may also be inferred that if r be a root of an equation, $x-r$ *will be a factor of the equation* when the equation is written with all its terms on one side.

If r and r' represent the roots of the general quadratic equation,

$$x^2+px+q=0.$$

This equation may be written $(x-r)(x-r')=0$.

or, $$x^2-(r+r')x+rr'=0.$$

A form which shows that

the *sum* of the roots $=-p$,

and the *product* of the roots $=q$.

223. It will be seen from § 222 that an equation may be formed if its roots be known.

If the roots of an equation be -1 and $\frac{1}{4}$,

the equation will be $$(x+1)(x-\tfrac{1}{4})=0,$$

or, $$x^2+\frac{3x}{4}-\frac{1}{4}=0,$$

or, by multiplying by 4, $$4x^2+3x-1=0.$$

If the roots of an equation be 0, 1, 5,

the equation will be $$(x-0)(x-1)(x-5)=0;$$

that is, $$x(x-1)(x-5)=0,$$

or, $$x^3-6x^2+5x=0.$$

If x occur in *every* term, the equation will be satisfied by putting $x=0$, and may be reduced to an equation of the next lower degree by dividing every term by x.

224. By considering the roots of $x^2+px+q=0$,

namely, $$r=-\frac{p}{2}+\frac{1}{2}\sqrt{p^2-4q},$$

and $$r'=-\frac{p}{2}-\frac{1}{2}\sqrt{p^2-4q},$$

it will be seen that the *character of the roots* of an equation may be determined without solving it:

I. As the two roots have the same expression, $\sqrt{p^2-4q}$, *both* roots will be *real*, or *both* will be *imaginary*.

If both be real, *both* will be *rational* or *both surds*, according as p^2-4q *is* or *is not* a perfect square.

II. When p^2 *is greater than* $4q$, the two roots will be *real*, for then the expression p^2-4q is *positive*, and therefore $\sqrt{p^2-4q}$ can be found exactly or approximately.

Since also its value in one root is to be added to $-\frac{p}{2}$, and in the other to be subtracted from $-\frac{p}{2}$, the two roots will be *different in value*.

III. When p^2 *is equal to* $4q$, the roots will be *equal in value*.

IV. When p^2 *is less than* $4q$, the roots will be *imaginary*, for then the expression p^2-4q will be negative, and therefore $\sqrt{p^2-4q}$ represents the *even* root of a *negative* number, and is imaginary.

V. If $q\ (=r\times r')$ be *positive*, the roots, if real, will have the *same sign*, but opposite to that of p (since $r+r'=-p$).

But if q be *negative*, the roots will have *opposite signs*.

225. Determine by inspection the character of the roots of:

(1) $x^2-5x+6=0$.

In this equation p is -5, and q is 6.

$\therefore \sqrt{p^2-4q}=\sqrt{25-24}=1$.

$\therefore$ the roots will be *rational*, and *both positive*.

(2) $x^2+3x+1=0$.

In this equation, p is 3, and q is 1.

$\therefore \sqrt{p^2-4q}=\sqrt{9-4}=\sqrt{5}$.

$\therefore$ the roots will be *surds*, and *both negative*.

(3) $x^2+3x+4=0$.

In this equation p is 3, and q is 4.

$\therefore \sqrt{p^2-4q}=\sqrt{9-16}=\sqrt{-7}$.

$\therefore$ the roots will be *impossible*.

Ex. 84.

Form the equations whose roots are:

1. $2, 1$.
2. $7, -3$.
3. $\frac{1}{2}, \frac{1}{3}$.
4. $\frac{2}{3}, -\frac{3}{2}$.
5. $-5, -\frac{1}{2}$.
6. $-\frac{7}{5}, \frac{9}{7}$.
7. $3, -3, \frac{3}{4}, -\frac{3}{4}$.
8. $0, 1, 2, 3$.
9. $0, -\frac{1}{2}, \frac{3}{2}, -1$.
10. $a-2b,\ 3a+2b$.
11. $2a-b,\ b-3a$.
12. $a(a+1),\ 1-a$.

Determine by inspection the character of the roots of:

13. $x^2-7x+12=0$.
14. $x^2-7x-30=0$.
15. $x^2+4x-5=0$.
16. $5x^2+8=0$.
17. $x^2+4x+1=0$.
18. $x^2-2x+9=0$.
19. $3x^2-4x-4=0$.
20. $x^2+4x+4=0$.

226. Two other cases of the solution of equations *by completing the square* should be noticed.

I. When *any* two powers of x are involved, *one of which is the square of the other.*

II. When the *addition of a number* to an equation of the fourth degree will *make both sides complete squares.*

(1) Solve $8x^6+63x^3=8$.

In this equation the exponent 6 is the double of 3, hence x^6 is the square of x^3.

$$8x^6+63x^3=8$$
$$256x^6+(\ )+(63)^2=4225$$
$$16x^3+63=\pm 65$$
$$16x^3=2, \text{ or } -128$$
$$x^3=\tfrac{1}{8}, \text{ or } -8.$$

By taking cube root, $\quad x=\frac{1}{2}$, or -2.

The other roots of the equation are found by finding the remaining roots of the equations, $x^3=\frac{1}{8}$, and $x^3=-8$.

Since $\quad x^3=\frac{1}{8}$,

$$\therefore 8x^3-1=0.$$

Now, by § 230, $\quad 8x^3-1=(2x-1)(4x^2+2x+1)$.

$$\therefore (2x-1)(4x^2+2x+1)=0,$$

and is satisfied if $\quad 4x^2+2x+1=0$,

as well as if $\quad 2x-1=0$.

The solution of $\quad 4x^2+2x+1=0$

gives $\quad x=\frac{1}{4}(-1\pm\sqrt{-3})$.

Since $\quad x^3=-8$,

$$\therefore x^3+8=0.$$

Now, by § 230, $\quad x^3+8=(x+2)(x^2-2x+4)$.

$$\therefore (x+2)(x^2-2x+4)=0,$$

and is satisfied if $\quad x^2-2x-4=0$,

as well as if $\quad x+2=0$.

The solution of $\quad x^2-2x+4=0$

gives $\quad x=1\pm\sqrt{-3})$.

$$\therefore \text{the roots are } \tfrac{1}{2},\ -2,\ 1\pm\sqrt{-3},\ \tfrac{1}{4}(-1\pm\sqrt{-3}).$$

(2) Solve $x^4 - 10x^3 + 35x^2 - 50x + 24 = 0$.

Take the square root of the left side.

$$\begin{array}{r|l}
 & x^4 - 10x^3 + 35x^2 - 50x + 24 \;\big|\; x^2 - 5x + 5 \\
 & x^4 \\
2x^2 - 5x & -10x^3 + 35x^2 \\
 & -10x^3 + 25x^2 \\
\hline
2x^2 - 10x + 5 & 10x^2 - 50x + 24 \\
 & 10x^2 - 50x + 25 \\
\hline
 & \qquad\qquad -\ 1
\end{array}$$

It is now seen that if 1 were added, the square would be complete, and the equation would be

$$x^4 - 10x^3 + 35x^2 - 50x + 25 = 1.$$

Extract the square root, and the result is

$$x^2 - 5x + 5 = \pm 1.$$

That is, $x^2 - 5x = -4, \text{ or } -6,$

$$4x^2 - (\ ) + 25 = 9, \text{ or } 1,$$

$$2x - 5 = \pm 3, \text{ or } \pm 1,$$

$$2x = 8,\ 2,\ 6, \text{ or } 4.$$

$$\therefore x = 4,\ 1,\ 3, \text{ or } 2.$$

Ex. 85.

Find the *possible* roots of:

1. $x^6 + 7x^3 = 8$.
2. $x^4 - 5x^2 + 4 = 0$.
3. $37x^2 - 9 = 4x^4$.
4. $16x^8 = 17x^4 - 1$.
5. $(x^2 - 9)^2 = 3 + 11(x^2 - 2)$.
6. $19x^4 + 216x^7 = x$.
7. $4x^4 - 20x^3 + 23x^2 + 5x = 6$.
8. $x^4 - 4x^3 - 10x^2 + 28x - 15 = 0$.
9. $x^4 - 2x^3 - 13x^2 + 14x = -24$.
10. $(x^2 - 1)(x^2 - 2) + (x^2 - 3)(x^2 - 4) = x^4 + 5$.

PROBLEMS INVOLVING QUADRATICS.

227. Problems which involve quadratic equations have apparently *two* solutions, as a quadratic has *two* roots. Sometimes both will be solutions; but generally one only will be a solution, and the other be inconsistent with the conditions of the problem. No difficulty will be found in selecting the result which belongs to the problem, and sometimes a change may be made in the statement of a problem so as to form a new problem corresponding to the solution which was inapplicable to the original problem.

(1) The sum of the squares of two consecutive numbers is 481. Find the numbers.

Let $x =$ one number,
and $x + 1 =$ the other.
Then $x^2 + (x+1)^2 = 481$,
or $2x^2 + 2x + 1 = 481$.

The solution of which gives, $x = 15$, or -16.

The positive root 15 gives for the numbers, 15 and 16.

The negative root -16 is inapplicable to the problem, as *consecutive numbers* are understood to be integers which follow one another in the common scale, 1, 2, 3, 4.....

(2) What is the price of eggs per dozen when 2 more in a shilling's worth lowers the price 1 penny per dozen?

Let $x =$ number of eggs for a shilling.

Then, $\frac{1}{x} =$ cost of 1 egg in shillings,

and $\frac{12}{x} =$ cost of 1 dozen in shillings.

But, if $x + 2 =$ number of eggs for a shilling,

$\frac{12}{x+2} =$ cost of 1 dozen in shillings.

$$\therefore \frac{12}{x} - \frac{12}{x+2} = \frac{1}{12} \text{ (1 penny being } \tfrac{1}{12} \text{ of a shilling).}$$

The solution of which gives $x = 16$, or -18.

And, if 16 eggs cost a shilling, 1 dozen will cost $\frac{12}{16}$ of a shilling, or 9 pence.

Therefore, the price of the eggs is 9 pence per dozen.

If the problem be changed so as to read: What is the price of eggs per dozen when two *less* in a shilling's worth *raises* the price 1 penny per dozen? the algebraic statement will be

$$\frac{12}{x-2} - \frac{12}{x} = \frac{1}{12}.$$

The solution of which gives $x = 18$, or -16.

Hence, the number 18, which had a negative sign and was inapplicable in the original problem, is here the true result.

Ex. 86.

1. The sum of the squares of three consecutive numbers is 365. Find the numbers.

2. Three times the product of two consecutive numbers exceeds four times their sum by 8. Find the numbers.

3. The product of three consecutive numbers is equal to three times the middle number. Find the numbers.

4. A boy bought a number of apples for 16 cents. Had he bought 4 more for the same money, he would have paid $\frac{1}{3}$ of a cent less for each apple. How many did he buy?

5. For building 108 rods of stone wall, 6 days less would have been required if 3 rods more a day had been built. How many rods a day were built?

6. A merchant bought some pieces of silk for \$900. Had he bought 3 pieces more for the same money, he would have paid \$15 less for each piece. How many did he buy?

7. A merchant bought some pieces of cloth for \$168.75. He sold the cloth for \$12 a piece, and gained as much as 1 piece cost him. Find the price of each piece.

8. The area of a square may be doubled by increasing its length by 6 inches and its breadth by 4 inches. Determine its side.

9. The length of a rectangular field exceeds the breadth by 1 yard, and the area is 3 acres. Determine its dimensions.

10. There are three lines of which two are each $\frac{4}{7}$ of the third, and the sum of the squares described on them is equal to a square yard. Determine the lengths of the lines in inches.

11. A grass plot 9 yards long and 6 yards broad has a path round it. The area of the path is equal to that of the plot. Determine the width of the path.

12. A can do some work in 9 hours less time than B can do it, and together they can do it in 20 hours. How long will it take each alone to do it?

13. A vessel which has two pipes can be filled in 2 hours less time by one than by the other, and by both together in 2 hours 55 minutes. How long will it take each pipe alone to fill the vessel?

14. A number is expressed by two digits, one of which is the square of the other, and when 54 is added, its digits are interchanged. Find the number.

15. A merchant expended a certain sum of money in goods, which he sold again for \$24, and lost as much per cent as the goods cost him. How much did he pay for the goods?

CHAPTER XV.

SIMULTANEOUS QUADRATIC EQUATIONS.

228. Quadratic equations involving *two* unknown quantities require different methods for their solution, according to the *form* of the equations.

229. CASE I. When from one of the equations the value of one of the unknown quantities can be found in terms of the other, and this value *substituted* in the other equation.

Ex. Solve
$$\left.\begin{array}{l}3x^2 - 2xy = 5 \\ x - y = 2\end{array}\right\} \quad \begin{array}{r}(1) \\ (2)\end{array}$$

Transpose x in (2), $y = x - 2.$
Substitute in (1), $3x^2 - 2x(x - 2) = 5.$
The solution of which gives $x = 1 \text{ or } -5.$
$\therefore y = -1 \text{ or } -7.$

Special methods often give more elegant solutions of examples than the *general* method by *substitution*.

I. *When equations have the form,* $x \pm y = a$, *and* $xy = b$; $x^2 \pm y^2 = a$, *and* $xy = b$; *or,* $x \pm y = a$, *and* $x^2 + y^2 = b$.

(1) Solve
$$\left.\begin{array}{l}x + y = 40 \\ xy = 300\end{array}\right\} \quad \begin{array}{r}(1) \\ (2)\end{array}$$

Square (1) $x^2 + 2xy + y^2 = 1600.$ (3)
Multiply (2) by 4, $4xy = 1200.$ (4)

Subtract (4) from (3), $x^2 - 2xy + y^2 = 400.$
Extract root of each side, $x - y = \pm 20.$ (6)
Add (1) and (6), $2x = 60 \text{ or } 20.$
$\therefore x = 30 \text{ or } 10.$
Subtract (6) from (1), $2y = 20 \text{ or } 60.$
$\therefore y = 10 \text{ or } 30.$

(2) Solve
$$\left.\begin{aligned} x - y &= 4 \\ x^2 + y^2 &= 40 \end{aligned}\right\} \qquad \begin{matrix}(1)\\(2)\end{matrix}$$

Square (1), $x^2 - 2xy + y^2 = 16.$ (3)
Subtract (2) from (3), $-2xy = -24.$ (4)
Subtract (4) from (2), $x^2 + 2xy + y^2 = 64.$
Extract the root, $x + y = \pm 8.$ (5)
By combining (5) and (1), $x = 6$ or $-2.$
$y = 2$ or $-6.$

(3) Solve
$$\left.\begin{aligned} \frac{1}{x} + \frac{1}{y} &= \frac{9}{20} \\ \frac{1}{x^2} + \frac{1}{y^2} &= \frac{41}{400} \end{aligned}\right\} \qquad \begin{matrix}(1)\\(2)\end{matrix}$$

Square (1), $\frac{1}{x^2} + \frac{2}{xy} + \frac{1}{y^2} = \frac{81}{400}.$ (3)

Subtract (2) from (3), $\frac{2}{xy} = \frac{40}{400}.$ (4)

Subtract (4) from (2), $\frac{1}{x^2} - \frac{2}{xy} + \frac{1}{y^2} = \frac{1}{400}.$

Extract the root, $\frac{1}{x} - \frac{1}{y} = \pm \frac{1}{20}.$ (5)

By combining (1) and (5), $x = 4$ or $5.$
$y = 5$ or $4.$

II. *When one equation may be simplified by dividing it by the other.*

(4) Solve
$$\left.\begin{aligned} x^3 + y^3 &= 91 \\ x + y &= 7 \end{aligned}\right\} \qquad \begin{matrix}(1)\\(2)\end{matrix}$$

Divide (1) by (2), $x^2 - xy + y^2 = 13.$ (3)
Square (2), $x^2 + 2xy + y^2 = 49.$ (4)
Subtract (3) from (4), $3xy = 36.$
Divide by -3, $-xy = -12.$ (5)
Add (5) and (3), $x^2 - 2xy + y^2 = 1.$
Extract the root, $x - y = \pm 1.$ (6)
By combining (6) and (2), $x = 4$ or $3.$
$y = 3$ or $4.$

Ex. 87.

Solve:

1. $\left.\begin{array}{l} x+y=13 \\ xy=36 \end{array}\right\}$

2. $\left.\begin{array}{l} x+y=29 \\ xy=100 \end{array}\right\}$

3. $\left.\begin{array}{l} x-y=19 \\ xy=66 \end{array}\right\}$

4. $\left.\begin{array}{l} x-y=45 \\ xy=250 \end{array}\right\}$

5. $\left.\begin{array}{l} x-y=10 \\ x^2+y^2=178 \end{array}\right\}$

6. $\left.\begin{array}{l} x-y=14 \\ x^2+y^2=436 \end{array}\right\}$

7. $\left.\begin{array}{l} x+y=12 \\ x^2+y^2=104 \end{array}\right\}$

8. $\left.\begin{array}{l} \dfrac{1}{x}+\dfrac{1}{y}=\dfrac{3}{4} \\ \dfrac{1}{x^2}+\dfrac{1}{y^2}=\dfrac{5}{16} \end{array}\right\}$

9. $\left.\begin{array}{l} \dfrac{1}{x}+\dfrac{1}{y}=5. \\ \dfrac{1}{x^2}+\dfrac{1}{y^2}=13 \end{array}\right\}$

10. $\left.\begin{array}{l} 7x^2-8xy=159 \\ 5x+2y=7 \end{array}\right\}$

11. $\left.\begin{array}{l} x+y=49 \\ x^2+y^2=1681 \end{array}\right\}$

12. $\left.\begin{array}{l} x^3+y^3=341 \\ x+y=11 \end{array}\right\}$

13. $\left.\begin{array}{l} x^3+y^3=1008 \\ x+y=12 \end{array}\right\}$

14. $\left.\begin{array}{l} x^3-y^3=98 \\ x-y=2 \end{array}\right\}$

15. $\left.\begin{array}{l} x^3-y^3=279 \\ x-y=3 \end{array}\right\}$

16. $\left.\begin{array}{l} x-3y=1 \\ xy+y^2=5 \end{array}\right\}$

17. $\left.\begin{array}{l} 4y=5x+1 \\ 2xy=33-x^2 \end{array}\right\}$

18. $\left.\begin{array}{l} \dfrac{1}{x}-\dfrac{1}{y}=3 \\ \dfrac{1}{x^2}-\dfrac{1}{y^2}=21 \end{array}\right\}$

19. $\left.\begin{array}{l} \dfrac{1}{x}-\dfrac{1}{y}=2\frac{1}{2} \\ \dfrac{1}{x^2}-\dfrac{1}{y^2}=8\frac{3}{4} \end{array}\right\}$

20. $\left.\begin{array}{l} x^2-2xy-y^2=1 \\ x+y=2 \end{array}\right\}$

230. Case II. When each of the two equations is *homogeneous* and of the *second degree*.

Ex. Solve
$$\left.\begin{aligned}2y^2-4xy+3x^2&=17\\ y^2-x^2&=16\end{aligned}\right\}\qquad\begin{aligned}(1)\\(2)\end{aligned}$$

Let $y = vx$, and substitute vx for y in both equations.

From (1), $2v^2x^2 - 4vx^2 + 3x^2 = 17.$

$$\therefore\ x^2 = \frac{17}{2v^2-4v+3}.$$

From (2), $v^2x^2 - x^2 = 16.$

$$\therefore\ x^2 = \frac{16}{v^2-1}.$$

Equate the values of x^2,

$$\frac{17}{2v^2-4v+3} = \frac{16}{v^2-1}$$
$$32v^2 - 64v + 48 = 17v^2 - 17$$
$$15v^2 - 64v = -65.$$

The solution gives $v = \frac{5}{3}$ or $\frac{13}{5}.$

Substitute the value of v in $x^2 = \frac{16}{v^2-1},$

then $x^2 = 9$ or $\frac{25}{9}.$

$$\therefore\ x = \pm 3 \text{ or } \pm\frac{5}{3},$$

and $y = vx = \pm 5$ or $\pm\frac{13}{3}.$

Ex. 88.

Solve:

1. $\left.\begin{aligned}x^2+xy+2y^2&=74\\ 2x^2+2xy+y^2&=73\end{aligned}\right\}$

2. $\left.\begin{aligned}x^2+xy+4y^2&=6\\ 3x^2+8y^2&=14\end{aligned}\right\}$

3. $\left.\begin{aligned}x^2-xy+y^2&=21\\ y^2-2xy&=-15\end{aligned}\right\}$

4. $\left.\begin{aligned}x^2-4y^2-9&=0\\ xy+2y^2-3&=0\end{aligned}\right\}$

5. $\left.\begin{aligned}x^2-xy-35&=0\\ xy+y^2-18&=0\end{aligned}\right\}$

6. $\left.\begin{aligned}x^2+xy+2y^2&=44\\ 2x^2-xy+y^2&=16\end{aligned}\right\}$

231. CASE III. When the two equations are *symmetrical* with respect to x and y; that is, when they have x and y similarly involved in them.

Thus, the expressions $2x^3 + 3x^2y^2 + 2y^3$, $2xy - 3x - 3y + 1$, $x^4 - 3x^2y - 3xy^2 + y^4$ are symmetrical expressions.

(1) Solve

$$\left.\begin{aligned} x^3 + y^3 &= 18xy \quad (1)\\ x + y &= 12 \quad (2)\end{aligned}\right\}$$

Put $u + v$ for x, and $u - v$ for y, in (1) and (2).

(1) becomes $(u+v)^3 + (u-v)^3 = 18(u+v)(u-v)$,

or $u^3 + 3uv^2 = 9(u^2 - v^2)$. (3)

(2) becomes $(u+v) + (u-v) = 12$,

or $2u = 12$.

$\therefore u = 6$.

Substitute 6 for u in (3).

(3) becomes $216 + 18v^2 = 9(36 - v^2)$,

whence $v^2 = 4$.

$\therefore v = \pm 2$.

$\therefore x = u + v = 6 \pm 2 = 8$ or 4,

and $y = u - v = 6 \mp 2 = 4$ or 8.

(2) Solve

$$\left.\begin{aligned} x + y &= 8 \quad (1)\\ x^4 + y^4 &= 706 \quad (2)\end{aligned}\right\}$$

Put $u + v$ for x, and $u - v$ for y, in (1) and (2).

(1) becomes $(u+v) + (u-v) = 8$.

$\therefore u = 4$.

(2) becomes $u^4 + 6u^2v^2 + v^4 = 353$. (3)

Substitute 4 for u in (3),

$256 + 96v^2 + v^4 = 353$,

or $v^4 + 96v^2 = 97$. (4)

The solution of (4) gives $v = \pm 1$, or $\pm\sqrt{-97}$.

Taking the possible values of v, $x = 5$ or 3, and $y = 3$ or 5.

Ex. 89.

Solve:

1. $\left.\begin{aligned} 4xy &= 96 - x^2y^2\\ x + y &= 6\end{aligned}\right\}$

2. $\left.\begin{aligned} x^2 + y^2 &= 18 - x - y\\ xy &= 6\end{aligned}\right\}$

3. $\left.\begin{aligned} 2(x^2+y^2) &= 5xy \\ 4(x-y) &= xy \end{aligned}\right\}$

4. $\left.\begin{aligned} 4(x+y) &= 3xy \\ x+y+x^2+y^2 &= 26 \end{aligned}\right\}$

5. $\left.\begin{aligned} 4x^2+xy+4y^2 &= 58 \\ 5x^2+5y^2 &= 65 \end{aligned}\right\}$

6. $\left.\begin{aligned} xy(x+y) &= 30 \\ x^3+y^3 &= 35 \end{aligned}\right\}$

232. The preceding cases are *general methods* for the solution of equations which belong to the kinds referred to; often, however, in the solution of these and other kinds of simultaneous equations involving quadratics, a little ingenuity will suggest some step by which the roots may easily be found.

Ex. 90.

Solve:

1. $\left.\begin{aligned} x-y &= 7 \\ x^2+xy+y^2 &= 13 \end{aligned}\right\}$

2. $\left.\begin{aligned} xy-12 &= 0 \\ x-2y &= 5 \end{aligned}\right\}$

3. $\left.\begin{aligned} xy-7 &= 0 \\ x^2+y^2 &= 50 \end{aligned}\right\}$

4. $\left.\begin{aligned} 2x-5y &= 9 \\ x^2-xy+y^2 &= 7 \end{aligned}\right\}$

5. $\left.\begin{aligned} x-y &= 9 \\ xy+8 &= 0 \end{aligned}\right\}$

6. $\left.\begin{aligned} 5x-7y &= 0 \\ 5x^2-\frac{13xy}{4} &= 4-7y^2 \end{aligned}\right\}$

7. $\left.\begin{aligned} x-y &= 1 \\ x^2+y^2 &= 8\tfrac{1}{2} \end{aligned}\right\}$

8. $\left.\begin{aligned} x^2-xy+y^2 &= 48 \\ x-y-8 &= 0 \end{aligned}\right\}$

9. $\left.\begin{aligned} x^2+3xy+y^2 &= 1 \\ 3x^2+xy+3y^2 &= 13 \end{aligned}\right\}$

10. $\left.\begin{aligned} x+y &= a \\ 4xy-a^2 &= -4b^2 \end{aligned}\right\}$

11. $\left.\begin{aligned} x^2+9xy &= 340 \\ 7xy-y^2 &= 171 \end{aligned}\right\}$

12. $\left.\begin{aligned} x+y &= 6 \\ x^3+y^3 &= 72 \end{aligned}\right\}$

13. $\left.\begin{aligned} 3xy+2x+y &= 485 \\ 3x-2y &= 0 \end{aligned}\right\}$

14. $\left.\begin{aligned} x-y &= 1 \\ \frac{x}{y}+\frac{y}{x} &= 2\tfrac{1}{6} \end{aligned}\right\}$

15. $\left.\begin{aligned} x-y &= 1 \\ x^3-y^3 &= 19 \end{aligned}\right\}$

16. $\left.\begin{aligned} x^3+y^3 &= 2728 \\ x^2-xy+y^2 &= 124 \end{aligned}\right\}$

17. $\left.\begin{array}{l} x+y=a \\ x^2+y^2=b^2 \end{array}\right\}$

18. $\left.\begin{array}{l} x^2-y^2=0 \\ 3x^2-4xy+5y^2=9 \end{array}\right\}$

19. $\left.\begin{array}{l} x^2-xy+y^2=7 \\ x^4+x^2y^2+y^4=133 \end{array}\right\}$

20. $\left.\begin{array}{l} x^3-y^3=a^3 \\ x-y=a \end{array}\right\}$

21. $\left.\begin{array}{l} x^2+xy+y^2=37 \\ x^4+x^2y^2+y^4=481 \end{array}\right\}$

22. $\left.\begin{array}{l} \dfrac{x+y}{x-y}+\dfrac{x-y}{x+y}=\dfrac{89}{40} \\ 6x=20y+9 \end{array}\right\}$

Ex. 91.

1. If the length and breadth of a rectangle were each increased by 1, the area would be 48; if they were each diminished by 1, the area would be 24. Find the length and breadth.

2. The sum of the squares of the two digits of a number is 25, and the product of the digits is 12. Find the number.

3. The sum, the product, and the difference of the squares, of two numbers are all equal. Find the numbers.

NOTE. Represent the numbers by $x+y$ and $x-y$ respectively.

4. The difference of two numbers is $\frac{3}{8}$ of the greater, and the sum of their squares is 356. What are the numbers?

5. The numerator and denominator of one fraction are each greater by 1 than those of another, and the sum of the two fractions is $1\frac{5}{12}$; if the numerators were interchanged, the sum of the fractions would be $1\frac{1}{2}$. Find the fractions.

6. A man starts from the foot of a mountain to walk to its summit. His rate of walking during the second half of the distance is $\frac{1}{2}$ mile per hour less than his rate during the first half, and he reaches the summit in $5\frac{1}{2}$ hours. He descends in $3\frac{3}{4}$ hours, by walking 1 mile more per hour than during the first half of the ascent. Find the distance to the top and the rates of walking.

NOTE. Let $2x =$ the distance, and y miles per hour $=$ the rate at first.

Then $\frac{x}{y}+\frac{x}{y-\frac{1}{2}}=5\frac{1}{2}$ hours, and $\frac{2x}{y+1}=3\frac{3}{4}$ hours.

7. The sum of two numbers which are formed by the same two digits in reverse order is $\frac{5}{18}$ of their difference; and the difference of the squares of the numbers is 3960. Determine the numbers.

8. The hypotenuse of a right triangle is 20, and the area of the triangle is 96. Determine the sides.

NOTE. The square on the hypotenuse $=$ sum of the squares on the sides; and the area of a right triangle $= \frac{1}{2}$ product of sides.

9. Two boys run in opposite directions round a rectangular field the area of which is an acre; they start from one corner, and meet 13 yards from the opposite corner; and the rate of one is $\frac{5}{6}$ of the rate of the other. Determine the dimensions of the field.

10. The fore wheel of a carriage turns in a mile 132 times more than the hind wheel; but if the circumferences were each increased by 2 feet, it would turn only 88 times more. Find the circumference of each.

CHAPTER XVI.

THEORY OF EXPONENTS.

233. The expression a^n, when n is a positive integer, has been defined as the product of n equal factors each equal to a. § 24.

And it has been shown that $a^m \times a^n = a^{m+n}$. § 66.

That $a^m \div a^n = a^{m-n}$, if m be greater than n; § 93.

or $\frac{1}{a^{n-m}}$, if m be less than n. § 94.

And that $(a^m)^n = a^{mn}$. § 191.

Also, it is true that $a^n \times b^n = (ab)^n$; for

$(ab)^n = ab$ taken n times as a factor

$= a$ taken n times as a factor $\times$ b taken n times as a factor $= a^n \times b^n$.

234. Likewise, $\sqrt[n]{a}$, when n is a positive integer, has been defined as *one* of the n *equal factors* of a (§ 195); so that if $\sqrt[n]{a}$ be taken n times as a factor, the resulting product is a; that is, $(\sqrt[n]{a})^n = a$.

Again, the expression $\sqrt[n]{a^m}$ means that a is to be raised to the *mth* power, and the *nth* root of the result obtained.

And the expression $(\sqrt[n]{a})^m$ means that the *nth* root of a is to be taken, and the result raised to the *mth* power.

It will thus be seen that any proposition relating to roots and powers may be expressed by this method of notation. It is, however, *found convenient* to adopt another method of notation, in which fractional and negative exponents are used.

235. The meaning of a fractional exponent is at once suggested, by observing that the *division of an exponent*, when the resulting quotient is *integral*, is equivalent to extracting a root. Thus, a^3 is the square root of a^6, and 3, the exponent of a^3, is obtained by dividing the exponent of a^6 by 2.

If this division be indicated only, the square root of a^6 will be denoted by $a^{\frac{6}{2}}$, in which the *denominator* denotes the *root*, and the *numerator* the *power*. If the same meaning be given to an exponent when the division does not give an integral quotient, $a^{\frac{3}{2}}$ will represent the square root of the cube of a; and, in general, $a^{\frac{m}{n}}$, the *nth* root of the *mth* power of a. This, then, is the meaning that will be assigned to a fractional exponent, so that in a fractional exponent

236. *The numerator will indicate a power, and the denominator a root.*

237. The meaning of a negative exponent is suggested by observing that in a series of descending powers of a,

$$a^n \ldots\ldots a^5,\ a^4,\ a^3,\ a^2,\ a^1,$$

the subtraction of 1 from the exponent is equivalent to dividing by a; and if the operation be continued, the result is

$$a^0,\ a^{-1},\ a^{-2},\ a^{-3},\ a^{-4} \ldots\ldots a^{-n}.$$

Then, $$a^0 = \frac{a}{a} = 1;\ a^{-1} = 1 \div a = \frac{1}{a};$$

$$a^{-2} = \frac{1}{a} \div a = \frac{1}{a^2};\ a^{-n} = \frac{1}{a^n}.$$

This, then, is the meaning that will be assigned to a negative exponent, so that,

238. *A number with a negative exponent will denote the reciprocal of the number with the corresponding positive exponent.*

It may be easily shown that the laws which apply to positive integral exponents apply also to fractional and negative exponents.

239. To show that $a^{\frac{m}{n}} \times b^{\frac{m}{n}} = (ab)^{\frac{m}{n}}$:

$$\begin{aligned} a^{\frac{m}{n}} \times b^{\frac{m}{n}} &= \sqrt[n]{a^m} \times \sqrt[n]{b^m} \\ &= \sqrt[n]{a^m b^m} \\ &= \sqrt[n]{(ab)^m} \\ &= (ab)^{\frac{m}{n}} \qquad \text{(by definition).} \end{aligned}$$

Likewise, $\quad a^{\frac{1}{n}} \times b^{\frac{1}{n}} \times c^{\frac{1}{n}} = (abc)^{\frac{1}{n}}$, and so on.

240. To show that $(a^{\frac{1}{m}})^{\frac{1}{n}} = a^{\frac{1}{mn}}$:

Let $\qquad x = (a^{\frac{1}{m}})^{\frac{1}{n}}.$

Then $\qquad x^n = a^{\frac{1}{m}}$, and $x^{mn} = a.$

$$\therefore x = a^{\frac{1}{mn}}.$$

But $\qquad x = (a^{\frac{1}{m}})^{\frac{1}{n}} \qquad$ (by supposition).

$$\therefore (a^{\frac{1}{m}})^{\frac{1}{n}} = a^{\frac{1}{mn}}.$$

241. To show that $a^m \times a^{-n} = a^{m-n}$:

Now $\qquad a^m \times a^{-n} = a^m \times \dfrac{1}{a^n}$

$$= \frac{a^m}{a^n} = a^{m-n} \text{ if } m > n, \qquad \S\ 93.$$

or

$$= \frac{1}{a^{n-m}} \text{ if } m < n \qquad \S\ 94.$$

$$= a^{-(n-m)} \qquad \text{(by definition)}$$

$$= a^{m-n}.$$

242. In like manner the same laws may be shown to apply in every case.

243. Hence, whether m and n be *integral* or *fractional*, *positive* or *negative* :

I. $a^m \times a^n = a^{m+n}$. III. $(a^m)^n = a^{mn}$.

II. $a^m \div a^n = a^{m-n}$. IV. $a^m \times b^m = (ab)^m$.

Ex. 92.

Express with fractional exponents:

1. $\sqrt{x^3}$; $\sqrt[3]{x^2}$; $(\sqrt{x})^5$; $\sqrt[3]{a^4}$; $\sqrt[7]{a^6}$; $(\sqrt[3]{a})^7$; $\sqrt[6]{a^3b^2}$.

2. $\sqrt[3]{xy^2z^3}$; $\sqrt[5]{x^3y^2z^4}$; $\sqrt[7]{a^5b^6c^7}$; $5\sqrt{a^2bc^3x^4}$.

Express with radical signs:

3. $a^{\frac{2}{3}}$; $a^{\frac{1}{2}}b^{\frac{1}{6}}$; $4x^{\frac{1}{6}}y^{-\frac{5}{6}}$; $3x^{\frac{1}{3}}y^{-\frac{2}{3}}$.

Express with positive exponents:

4. a^{-2}; $3x^{-1}y^{-3}$; $6x^{-3}y$; x^4y^{-5}; $\dfrac{2a^{-1}x}{3^{-1}b^2y^{-3}}$.

Write in the form of integral expressions:

5. $\dfrac{3xy}{z^2}$; $\dfrac{z}{x^3y^4}$; $\dfrac{a}{bc}$; $\dfrac{c^2}{a^3b^{-2}}$; $\dfrac{x^{-\frac{1}{3}}}{y^{-\frac{2}{3}}}$; $\dfrac{x^{-2}}{y^{\frac{1}{3}}}$.

Simplify:

6. $a^{\frac{1}{2}} \times a^{\frac{1}{3}}$; $b^{\frac{1}{3}} \times b^{\frac{1}{6}}$; $c^{\frac{2}{3}} \times c^{\frac{1}{12}}$; $d^{\frac{3}{5}} \times d^{\frac{1}{15}}$.

7. $m^{\frac{1}{2}} \times m^{-\frac{1}{6}}$; $n^{\frac{3}{4}} \times n^{-\frac{1}{12}}$; $a^0 \times a^{\frac{1}{2}}$; $a^0 \times a^{-\frac{1}{2}}$.

8. $a^{\frac{1}{2}} \times \sqrt{a}$; $c^{-\frac{1}{2}} \times \sqrt{c}$; $y^{\frac{1}{4}} \times \sqrt[4]{y}$; $x^{\frac{5}{8}} \times \sqrt{x^{-1}}$.

9. $ab^{\frac{1}{2}}c \times a^{-\frac{1}{2}}bc^{\frac{1}{3}}$; $a^{\frac{2}{3}}b^{\frac{1}{2}}c^{-\frac{1}{4}} \times a^{\frac{1}{3}}b^{-\frac{1}{2}}c^{\frac{1}{2}}d$.

10. $x^{\frac{1}{3}}y^{\frac{2}{3}}z^{\frac{1}{6}} \times x^{-\frac{2}{3}}y^{-\frac{1}{2}}z^{-\frac{1}{2}}$; $x^{\frac{5}{6}}y^{\frac{1}{1}}z^{\frac{1}{2}} \times x^{-\frac{1}{6}}y^{-\frac{1}{2}}z^{-\frac{1}{2}}$.

11. $a^{\frac{1}{2}} \times a^{-\frac{1}{3}} \times a^{-\frac{1}{4}} \times a^{-\frac{1}{3}}$; $\left(\dfrac{ay}{x}\right)^{\frac{1}{2}} \times \left(\dfrac{bx}{y^2}\right)^{\frac{1}{3}} \times \left(\dfrac{y^2}{a^2b^2}\right)^{\frac{1}{4}}$.

12. $a^{\frac{1}{2}} \div a^{\frac{1}{3}}$; $c^{\frac{5}{8}} \div c^{\frac{1}{2}}$; $n^{\frac{7}{12}} \div n^{\frac{3}{4}}$; $a^{\frac{5}{6}} \div \sqrt[3]{a^2}$.

13. $(a^6)^{\frac{1}{2}} \div (a^6)^{\frac{2}{3}}$; $(c^{-\frac{1}{2}})^{\frac{2}{5}}$; $(m^{-\frac{1}{2}})^4$; $(n^{\frac{1}{3}})^{-3}$; $(x^{\frac{3}{4}})^{\frac{4}{3}}$.

14. $(p^{-\frac{3}{4}})^{-\frac{2}{3}}$; $(q^{\frac{4}{5}})^{-\frac{1}{2}}$; $(x^{-\frac{3}{8}}y^{\frac{9}{4}})^{-\frac{4}{9}}$; $(a^{\frac{2}{3}} \times a^{\frac{4}{7}})^{-\frac{14}{15}}$.

15. $(4a^{-\frac{2}{3}})^{-\frac{3}{2}}$; $(27b^{-3})^{-\frac{2}{3}}$; $(64c^{16})^{-\frac{5}{6}}$; $(32c^{-10})^{\frac{2}{5}}$.

16. $\left(\dfrac{16a^{-4}}{81b^3}\right)^{-\frac{3}{4}}$; $\left(\dfrac{9a^4}{16b^{-3}}\right)^{-\frac{3}{2}}$; $(3^{\frac{3}{8}}a^{-3})^{-\frac{2}{3}}$; $(\frac{256}{625})^{-\frac{3}{4}}$.

244. The laws that apply to the exponents of simple expressions also apply to the exponents of compound expressions.

(1) Multiply $y^{\frac{3}{4}} + y^{\frac{1}{2}} + y^{\frac{1}{4}} + 1$ by $y^{\frac{1}{4}} - 1$.

$$\begin{array}{l}
y^{\frac{3}{4}} + y^{\frac{1}{2}} + y^{\frac{1}{4}} + 1 \\
\underline{y^{\frac{1}{4}} - 1} \\
y + y^{\frac{3}{4}} + y^{\frac{1}{2}} + y^{\frac{1}{4}} \\
\underline{\quad - y^{\frac{3}{4}} - y^{\frac{1}{2}} - y^{\frac{1}{4}} - 1} \\
y \qquad\qquad\qquad\quad - 1
\end{array}$$

$y - 1$. *Ans.*

(2) Divide $x^{\frac{2}{3}} + x^{\frac{1}{3}} - 12$ by $x^{\frac{1}{3}} - 3$.

$$\begin{array}{l}
x^{\frac{2}{3}} + \quad x^{\frac{1}{3}} - 12 \left(\dfrac{x^{\frac{1}{3}} - 3}{x^{\frac{1}{3}} + 4} \right. \\
\underline{x^{\frac{2}{3}} - 3x^{\frac{1}{3}}} \\
\qquad 4x^{\frac{1}{3}} - 12 \\
\qquad 4x^{\frac{1}{3}} - 12
\end{array}$$

$x^{\frac{1}{3}} + 4$. *Ans.*

Ex. 93.

Multiply:

1. $x^{2p} + x^p y^p + y^{2p}$ by $x^{2p} - x^p y^p + y^{2p}$.

2. $x^{mn-n} - y^n$ by $x^n + y^{mn-n}$.

3. $x^{\frac{2}{3}} - 2x^{\frac{1}{3}} + 1$ by $x^{\frac{1}{3}} - 1$.

4. $8a^{\frac{3}{7}} + 4a^{\frac{2}{7}}b^{\frac{1}{7}} + 5a^{\frac{1}{7}}b^{\frac{2}{7}} + 9b^{\frac{3}{7}}$ by $2a^{\frac{4}{7}} - b^{\frac{4}{7}}$.

5. $1 + ab^{-1} + a^2b^{-2}$ by $1 - ab^{-1} + a^2b^{-2}$.

6. $a^2b^{-2} + 2 + a^{-2}b^2$ by $a^2b^{-2} - 2 - a^{-2}b^2$.

7. $4x^{-3} + 3x^{-2} + 2x^{-1} + 1$ by $x^{-2} - x^{-1} + 1$.

Divide:

8. $x^{4n} - y^{4n}$ by $x^n - y^n$.

9. $x + y + z - 3x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}}$ by $x^{\frac{1}{3}} + y^{\frac{1}{3}} + z^{\frac{1}{3}}$.

10. $x + y$ by $x^{\frac{4}{5}} - x^{\frac{3}{5}}y^{\frac{1}{5}} + x^{\frac{2}{5}}y^{\frac{2}{5}} - x^{\frac{1}{5}}y^{\frac{3}{5}} + y^{\frac{4}{5}}$.

11. $x^2y^{-2} + 2 + x^{-2}y^2$ by $xy^{-1} + x^{-1}y$.

12. $a^{-4} + a^{-2}b^{-2} + b^{-4}$ by $a^{-2} - a^{-1}b^{-1} + b^{-2}$.

Find the squares of:

13. $4ab^{-1}$; $a^{\frac{1}{2}} - b^{\frac{1}{2}}$; $a + a^{-1}$; $2a^{\frac{1}{2}}b^{\frac{1}{3}} - a^{-\frac{1}{2}}b^{\frac{2}{3}}$.

If $a = 4$, $b = 2$, $c = 1$, find the values of:

14. $a^{\frac{1}{2}}b$; $5ab^{-1}$; $2(ab)^{\frac{1}{3}}$; $a^{-\frac{1}{2}}b^{-1}c^{\frac{2}{3}}$; $12a^{-2}b^{-3}$.

15. Expand $(a^{\frac{1}{2}} - b^{\frac{1}{3}})^3$; $(2x^{-1} + x)^4$; $(ab^{-1} - by^{-1})^6$.

Extract the square root of:

16. $9x^{-4} - 18x^{-3}y^{\frac{1}{2}} + 15x^{-2}y - 6x^{-1}y^{\frac{3}{2}} + y^2$.

Extract the cube root of:

17. $8x^3 + 12x^2 - 30x - 35 + 45x^{-1} + 27x^{-2} - 27x^{-3}$.

Resolve into prime factors with fractional exponents:

18. $\sqrt[3]{12}$, $\sqrt[4]{72}$, $\sqrt[6]{96}$, $\sqrt[8]{64}$; and find their product.

Simplify:

19. $\{(x^{5ab})^3 \times (x^{5b})^{-2}\}^{\frac{1}{3a-2}}$. 20. $(x^{18a} \times x^{-12})^{\frac{1}{3a-2}}$.

21. $3(a^{\frac{1}{2}} + b^{\frac{1}{2}})^2 - 4(a^{\frac{1}{2}} + b^{\frac{1}{2}})(a^{\frac{1}{2}} - b^{\frac{1}{2}}) + (a^{\frac{1}{2}} - 2b^{\frac{1}{2}})^2$.

22. $\{(a^m)^{m - \frac{1}{m}}\}^{\frac{1}{m+1}}$.

Radical Expressions.

245. An indicated root that cannot be exactly obtained is called a **surd, or irrational number.** An indicated root that can be exactly obtained is said to have the *form* of a surd.

246. The required root shows the **order** of a surd; and surds are named *quadratic*, *cubic*, *biquadratic*, according as the *second*, *third*, or *fourth* roots are required.

247. The product of a rational factor and a surd factor is called a *mixed surd;* as, $3\sqrt{2}$, $b\sqrt{a}$.

248. When there is no rational factor outside of the radical sign, the surd is said to be *entire;* as, $\sqrt{2}$, $\sqrt{a}$.

249. Since $\sqrt[m]{a} \times \sqrt[m]{b} \times \sqrt[m]{c} = \sqrt[m]{abc}$, the product of two or more surds of the same order will be a radical expression of the same order consisting of the product of the numbers under the radical signs.

250. In like manner, $\sqrt{a^2b} = \sqrt{a^2} \times \sqrt{b} = a\sqrt{b}$. That is,

A factor under the radical sign whose root can be taken, may, by having the root taken, be removed from under the radical sign.

251. Conversely, since $a\sqrt{b} = \sqrt{a^2b}$,

A factor outside the radical sign may be raised to the corresponding power, and placed under it.

Again: $$\sqrt{\frac{a}{b^2}} = \sqrt{a \times \frac{1}{b^2}} = \frac{1}{b}\sqrt{a};$$

and $$\sqrt{\frac{a}{b}} = \sqrt{\frac{ab}{b^2}} = \sqrt{ab \times \frac{1}{b^2}} = \frac{1}{b}\sqrt{ab}.$$

252. A surd is in its *simplest form* when the expression under the radical sign is *integral* and *as small as possible.*

253. Surds which, when reduced to the simplest form, have the *same surd factor*, are said to be similar.

Simplify:

$$\sqrt{50};\quad \sqrt[3]{108};\quad \sqrt[5]{7x^2y^7};\quad \sqrt{\frac{7}{12}};\quad \sqrt[4]{\frac{5a}{2b^3c^2}};\quad \sqrt[3]{296352}.$$

(1) $\sqrt{50}=\sqrt{25\times 2}=5\sqrt{2}.$

(2) $\sqrt[3]{108}=\sqrt[3]{27\times 4}=3\sqrt[3]{4}.$

(3) $\sqrt[5]{7x^2y^7}=\sqrt[5]{7x^2y^2\times y^5}=y\sqrt[5]{7x^2y^2}.$

(4) $\sqrt{\frac{7}{12}}=\sqrt{\frac{7}{4\times 3}}=\sqrt{\frac{7\times 3}{4\times 9}}=\sqrt{21\times\frac{1}{4\times 9}}=\frac{1}{6}\sqrt{21}.$

(5) $\sqrt[4]{\frac{5a}{2b^3c^2}}=\sqrt[4]{\frac{40\,abc^2}{16\,b^4c^4}}=\frac{1}{2bc}\sqrt[4]{40\,abc^2}.$

(6) $\sqrt[3]{296352}.$

$$\begin{array}{l|r}
2^3 & 296352 \\ \hline
2^2 & 37044 \\ \hline
3^2 & 9261 \\ \hline
3 & 1029 \\ \hline
7 & 343 \\ \hline
7 & 49 \\ \hline
 & 7
\end{array}$$

Hence, $296352=2^5\times 3^3\times 7^3.$

$$\begin{aligned}\therefore\ \sqrt[3]{296352}&=\sqrt[3]{7^3}\times\sqrt[3]{3^3}\times\sqrt[3]{2^5}\\ &=7\times 3\times 2\sqrt[3]{2^2}\\ &=42\sqrt[3]{4}.\ \textit{Ans.}\end{aligned}$$

In simplifying numerical expressions under the radical sign, the method employed in (6) may be used with advantage when the factor whose root can be taken is not readily determined by inspection.

Ex. 94.

Express as entire surds:

1. $3\sqrt{5}$; $3\sqrt{21}$; $5\sqrt{32}$; $a^2b\sqrt{bc}$; $x\sqrt[3]{x^2y^2}$.

2. $3y^2\sqrt[4]{x^3y}$; $2x\sqrt[5]{xy}$; $a^3\sqrt[4]{a^3b^2}$; $3c^2\sqrt[3]{abc}$; $5abc\sqrt{abc^{-1}}$.

3. $\frac{2}{7}\sqrt{\frac{91}{8}}$; $16\sqrt{\frac{275}{288}}$; $(x+y)\sqrt{\frac{xy}{x^2+2xy+y^2}}$.

Express as mixed surds:

4. $\sqrt{x^2y^4z}$; $\sqrt{8a^3b}$; $\sqrt[3]{54a^4x^2y^3}$; $\sqrt{24}$; $\sqrt{125a^4d^3}$.

5. $\sqrt[3]{1000a}$; $\sqrt[3]{160x^4y^7}$; $\sqrt[3]{108m^9n^{10}}$; $\sqrt[3]{1372a^{15}b^{16}}$.

6. $\sqrt[3]{a^4-3a^3b+3a^2b^2-ab^3}$; $\sqrt{50a^2-100ab+50b^2}$.

Simplify:

7. $2\sqrt[4]{80a^5b^2c^6}$; $7\sqrt{396x}$; $9\sqrt[3]{81x^2y^3z}$; $5\sqrt{726}$.

8. $\sqrt{\frac{5}{9}}$; $\sqrt{1\frac{11}{16}}$; $\sqrt{3\frac{1}{8}}$; $\frac{3}{5}\sqrt{90\frac{5}{8}}$; $2\sqrt[3]{\frac{1}{2}}$.

9. $\sqrt[3]{\dfrac{2xy^2}{z}}$; $\sqrt[3]{\frac{4}{25}}$; $\dfrac{a}{b}\sqrt[4]{\dfrac{b}{2a^3}}$; $\sqrt{\dfrac{3a^2bx}{4cy^3}}$.

10. $\dfrac{12}{\sqrt{5}}$; $\dfrac{2}{\sqrt{1701}}$; $\left(\dfrac{x^3y^2}{z^2}\right)\left(\dfrac{z^5}{x^5y^5}\right)^{\frac{1}{2}}$; $\left(\dfrac{a^3b^2}{c^4}\right)\left(\dfrac{c^9b^3}{a}\right)^{\frac{1}{3}}$.

11. $(ax)\times(b^2x)^{\frac{1}{2}}$; $(2a^2b^4)\times(b^2x^3)^{\frac{1}{3}}$; $5(3a^3b^4y)\times(a^5b^{-4}y^3)^{\frac{1}{4}}$.

12. Show that $\sqrt{20}$, $\sqrt{45}$, $\sqrt{\frac{4}{5}}$ are similar surds.

13. Show that $2\sqrt[3]{a^3b^2}$, $\sqrt[3]{8b^5}$, $\frac{1}{2}\sqrt[3]{\dfrac{a^6}{b}}$ are similar surds.

14. If $\sqrt{2}=1.414213$, find the values of

$$\sqrt{50};\ \tfrac{5}{2}\sqrt{288};\ \frac{1}{\sqrt{2}};\ \frac{3}{\sqrt{450}}.$$

254. Surds of *the same order* may be compared by expressing them as entire surds.

Ex. Compare $\frac{2}{3}\sqrt{7}$ and $\frac{3}{5}\sqrt{10}$.

$$\tfrac{2}{3}\sqrt{7} = \sqrt{\tfrac{28}{9}}.$$

$$\tfrac{3}{5}\sqrt{10} = \sqrt{\tfrac{18}{5}}.$$

$$\sqrt{\tfrac{28}{9}} = \sqrt{\tfrac{140}{45}}, \text{ and } \sqrt{\tfrac{18}{5}} = \sqrt{\tfrac{162}{45}}.$$

As $\sqrt{\frac{162}{45}}$ is greater than $\sqrt{\frac{140}{45}}$, $\frac{3}{5}\sqrt{10}$ is greater than $\frac{2}{3}\sqrt{7}$.

255. The product or quotient of two surds of *the same order* may be obtained by taking the product or quotient of the rational factors and the surd factors separately.

(1) $2\sqrt{5} \times 5\sqrt{7} = 10\sqrt{35}.$

(2) $9\sqrt{5} \div 3\sqrt{7} = 3\sqrt{\frac{5}{7}} = 3\sqrt{\frac{35}{49}} = \frac{3}{7}\sqrt{35}.$

Ex. 95.

1. Which is the greater, $3\sqrt{7}$ or $2\sqrt{15}$?
2. Arrange in order of magnitude $9\sqrt{3}$, $6\sqrt{7}$, $5\sqrt{10}$.
3. Arrange in order of magnitude $4\sqrt[3]{4}$, $3\sqrt[3]{5}$, $5\sqrt[3]{3}$.
4. Multiply $3\sqrt{2}$ by $4\sqrt{6}$; $\frac{3}{7}\sqrt{10}$ by $\frac{7}{10}\sqrt{15}$.
5. Multiply $5\sqrt{\frac{2}{7}}$ by $\frac{3}{7}\sqrt{162}$; $\frac{1}{2}\sqrt[3]{4}$ by $2\sqrt[3]{2}$.
6. Divide $2\sqrt{5}$ by $3\sqrt{15}$; $\frac{3}{8}\sqrt{21}$ by $\frac{9}{16}\sqrt{\frac{7}{20}}$.
7. Simplify $\frac{2}{5}\sqrt{3} \times \frac{4}{9}\sqrt{5} \div \frac{6}{7}\sqrt{2}$.
8. Simplify $\dfrac{2\sqrt{10}}{3\sqrt{27}} \times \dfrac{7\sqrt{48}}{5\sqrt{14}} \div \dfrac{4\sqrt{15}}{15\sqrt{21}}$.
9. Simplify $2\sqrt[3]{4} \times 5\sqrt[3]{32} \div \sqrt[3]{108}$.

256. The *order* of a surd may be changed by changing the *power* of the expression under the radical sign. Thus,

$$\sqrt{5} = \sqrt[4]{25};\quad \sqrt[3]{c} = \sqrt[6]{c^2}.$$

Conversely, $\sqrt[4]{25} = \sqrt{5}$; $\sqrt[6]{c^2} = \sqrt[3]{c}$;

or, in general, $\sqrt[mn]{c^n} = \sqrt[m]{c}.$

In this way, surds of *different orders* may be reduced to the *same order*, and may then be compared, multiplied, or divided.

(1) To compare $\sqrt{2}$ and $\sqrt[3]{3}$.

$$\sqrt{2} = 2^{\frac{1}{2}} = 2^{\frac{3}{6}} = \sqrt[6]{2^3} = \sqrt[6]{8};$$
$$\sqrt[3]{3} = 3^{\frac{1}{3}} = 3^{\frac{2}{6}} = \sqrt[6]{3^2} = \sqrt[6]{9}.$$

$\therefore \sqrt[3]{3}$ is greater than $\sqrt{2}$.

(2) To multiply $\sqrt[3]{4a}$ by $\sqrt{6x}$.

$$\sqrt[3]{4a} = (4a)^{\frac{1}{3}} = (4a)^{\frac{2}{6}} = \sqrt[6]{(4a)^2} = \sqrt[6]{16a^2};$$
$$\sqrt{6x} = (6x)^{\frac{1}{2}} = (6x)^{\frac{3}{6}} = \sqrt[6]{(6x)^3} = \sqrt[6]{216x^3}.$$
$$\begin{aligned}\therefore \sqrt[3]{4a} \times \sqrt{6x} &= \sqrt[6]{16a^2} \times \sqrt[6]{216x^3}\\ &= \sqrt[6]{16a^2 \times 216x^3}\\ &= \sqrt[6]{2^4a^2 \times 2^3 \times 3^3x^3}\\ &= \sqrt[6]{2^6 \times 2 \times 3^3a^2x^3}\\ &= 2\sqrt[6]{54a^2x^3}. \textit{ Ans.}\end{aligned}$$

(3) To divide $\sqrt[3]{3a}$ by $\sqrt{6b}$.

$$\sqrt[3]{3a} = (3a)^{\frac{1}{3}} = (3a)^{\frac{2}{6}} = \sqrt[6]{(3a)^2} = \sqrt[6]{9a^2};$$
$$\sqrt{6b} = (6b)^{\frac{1}{2}} = (6b)^{\frac{3}{6}} = \sqrt[6]{(6b)^3} = \sqrt[6]{216b^3}.$$
$$\begin{aligned}\therefore \sqrt[3]{3a} \div \sqrt{6b} &= \sqrt[6]{9a^2} \div \sqrt[6]{216b^3} = \sqrt[6]{\frac{9a^2}{216b^3}}\\ &= \sqrt[6]{\frac{a^2}{24b^3}} = \sqrt[6]{\frac{a^2}{2^3 \times 3b^3}}\\ &= \sqrt[6]{\frac{2^3 \times 3^5a^2b^3}{2^6 \times 3^6b^6}} = \frac{1}{6b}\sqrt[6]{1944a^2b^3}. \textit{ Ans.}\end{aligned}$$

Ex. 96.

Arrange in order of magnitude:

1. $2\sqrt[3]{3}$, $3\sqrt{2}$, $\frac{5}{2}\sqrt[4]{4}$.
2. $\sqrt{\frac{3}{5}}$, $\sqrt[3]{\frac{14}{15}}$.
3. $2\sqrt[3]{22}$, $3\sqrt[3]{7}$, $4\sqrt{2}$.
4. $3\sqrt{19}$, $5\sqrt[3]{2}$, $3\sqrt[3]{3}$.

Simplify:

5. $2\sqrt{ax} \times \sqrt[3]{3a^2b} \times \sqrt{2bx}$; $\sqrt[4]{a^3xy^3} \times \sqrt[5]{a^2xy}$.

6. $3(4ab^2)^{\frac{1}{3}} \div (2a^3b)^{\frac{1}{2}}$; $(2a^3b^2)^{\frac{1}{4}} \times (a^5b^3)^{\frac{1}{3}} \div (a^3b^5)^{\frac{1}{2}}$.

7. $(2ab)^{\frac{1}{2}} \times (3ab^2)^{\frac{1}{3}} \div (5ab^3)^{\frac{1}{6}}$; $4\sqrt{12} \div 2\sqrt{3}$.

8. $\left(\frac{ay}{x}\right)^{\frac{1}{2}} \times \left(\frac{bx}{y^2}\right)^{\frac{1}{3}} \div \left(\frac{y^2}{a^2b^2}\right)^{\frac{1}{4}}$.

9. $(7\sqrt{2} - 5\sqrt{6} - 3\sqrt{8} + 4\sqrt{20}) \times 3\sqrt{2}$.

10. $\sqrt{(\frac{16}{25})^7} \times \sqrt{(\frac{25}{64})^6}$; $\sqrt[3]{(4ab^2)^x} \times \sqrt[3]{(2a^2b)^x}$.

11. $(\sqrt[7]{a^3b})^3 \times (\sqrt[7]{a^3b^{12}})^4$; $a^{\frac{1}{2}}b^{-\frac{2}{3}}c^{\frac{3}{4}}d^{-\frac{5}{6}} \div a^{\frac{7}{8}}b^{-\frac{9}{10}}c^{-\frac{10}{11}}d^{\frac{11}{12}}$.

257. In the addition or subtraction of surds, each surd must be reduced to its simplest form; and, if the resulting surds be similar,

Add the rational factors, and to their sum annex the common surd factor.

If the resulting surds be not similar,

Connect them with their proper signs.

258. Operations with surds will be more easily performed if the arithmetical numbers contained in the surds be *expressed in their prime factors*, and if *fractional exponents* be used instead of radical signs.

(1) Simplify $\sqrt{27} + \sqrt{48} + \sqrt{147}$.

$$\sqrt{27} = (3^3)^{\frac{1}{2}} = 3 \times 3^{\frac{1}{2}} = 3\sqrt{3};$$
$$\sqrt{48} = (2^4 \times 3)^{\frac{1}{2}} = 2^2 \times 3^{\frac{1}{2}} = 4 \times 3^{\frac{1}{2}} = 4\sqrt{3};$$
$$\sqrt{147} = (7^2 \times 3)^{\frac{1}{2}} = 7 \times 3^{\frac{1}{2}} = 7\sqrt{3}.$$
$$\therefore \sqrt{27} + \sqrt{48} + \sqrt{147} = (3 + 4 + 7)\sqrt{3} = 14\sqrt{3}. \textit{ Ans.}$$

(2) Simplify $2\sqrt[3]{320} - 3\sqrt[3]{40}$.

$$2\sqrt[3]{320} = 2(2^6 \times 5)^{\frac{1}{3}} = 2 \times 2^2 \times 5^{\frac{1}{3}} = 8\sqrt[3]{5};$$
$$3\sqrt[3]{40} = 3(2^3 \times 5)^{\frac{1}{3}} = 3 \times 2 \times 5^{\frac{1}{3}} = 6\sqrt[3]{5}.$$
$$\therefore 2\sqrt[3]{320} - 3\sqrt[3]{40} = 8\sqrt[3]{5} - 6\sqrt[3]{5} = 2\sqrt[3]{5}. \quad \textit{Ans.}$$

(3) Find the square root of $\sqrt[3]{81}$.

The square root of $\sqrt[3]{81} = (81^{\frac{1}{3}})^{\frac{1}{2}} = 81^{\frac{1}{6}} = (3^4)^{\frac{1}{6}}$
$= 3^{\frac{2}{3}} = (3^2)^{\frac{1}{3}} = \sqrt[3]{9}.$

(4) Find the cube of $\frac{1}{2}\sqrt[6]{2}$.

The cube of $\frac{1}{2}\sqrt[6]{2} = (\frac{1}{2})^3 \times (2^{\frac{1}{6}})^3 = \frac{1}{8} \times 2^{\frac{1}{2}} = \frac{1}{8}\sqrt{2}.$

Ex. 97.

Simplify:

1. $\sqrt{27} + 2\sqrt{48} + 3\sqrt{108}$; $3\sqrt{1000} + 4\sqrt{50} + 12\sqrt{288}$.
2. $\sqrt[3]{128} + \sqrt[3]{686} + \sqrt[3]{16}$; $7\sqrt[3]{54} + 3\sqrt[3]{16} + \sqrt[3]{432}$.
3. $12\sqrt{72} - 3\sqrt{128}$; $7\sqrt[3]{81} - 3\sqrt[3]{1029}$.
4. $2\sqrt{3} + 3\sqrt{1\frac{1}{3}} - \sqrt{5\frac{1}{3}}$; $2\sqrt{\frac{5}{3}} + \sqrt{60} - \sqrt{15} - \sqrt{\frac{3}{5}}$.
5. $\sqrt{\dfrac{a^4c}{b^3}} - \sqrt{\dfrac{a^2c^3}{bd^2}} - \sqrt{\dfrac{a^2cd^2}{bm^2}}$; $3\sqrt{\frac{2}{5}} + 2\sqrt{\frac{1}{10}} - 4\sqrt{\frac{1}{40}}$.
6. $\sqrt{4a^3b} + \sqrt{25ab^3} - (a - 5b)\sqrt{ab}$.
7. $c\sqrt[5]{a^6b^7c^3} - a\sqrt[5]{ab^7c^8} + b\sqrt[5]{a^6b^2c^8}$.
8. $2\sqrt[3]{40} + 3\sqrt[3]{108} + \sqrt[3]{500} - \sqrt[3]{320} - 2\sqrt[3]{1372}$.
9. $(2\sqrt[6]{3a^4b})^3$; $(3\sqrt[6]{3})^2$. 10. $\left(\dfrac{a}{3}\sqrt{\dfrac{a}{3}}\right)^{\frac{1}{3}}$; $(\sqrt{27})^{\frac{1}{2}}$.
11. $(\sqrt[3]{81})^{\frac{1}{4}}$; $(\sqrt[4]{512})^{\frac{1}{3}}$; $(\sqrt[3]{256})^{-\frac{1}{4}}$; $\sqrt[12]{16}$; $\sqrt[12]{27}$.
12. $\sqrt[10]{4}$; $\sqrt[10]{36}$; $\sqrt[10]{32}$; $\sqrt[10]{243}$; $\sqrt[9]{125}$; $\sqrt[4]{49}$.
13. $\sqrt[9]{8x^6}$; $\sqrt[6]{9a^2b^4}$; $\sqrt[8]{16a^{12}}$; $\sqrt[5]{32a^{10}}$.

14. $(\sqrt[9]{8})^4$; $(\sqrt[9]{27})^4$; $(\sqrt[8]{64})^3$; $(\sqrt[3]{4})^2$.

15. $(a\sqrt[3]{a})^{-3}$; $(x\sqrt[3]{x})^{-\frac{1}{2}}$; $(p^2\sqrt{p})^{\frac{1}{5}}$; $(a^{-3}\sqrt[4]{a^{-8}})^{-\frac{1}{4}}$.

Find the square root of:

16. $x^{4m}+6x^{3m}y^n+11x^{2m}y^{2n}+6x^my^{3n}+y^{4n}$.

17. $1+4x^{-\frac{1}{3}}-2x^{-\frac{2}{3}}-4x^{-1}+25x^{-\frac{4}{3}}-24x^{-\frac{5}{3}}+16x^{-2}$.

259. If we wish to find the approximate value of $\frac{3}{\sqrt{2}}$, it will be less labor to multiply first both numerator and denominator by a factor that will render the denominator *rational;* in this case by $\sqrt{2}$. Thus,

$$\frac{3}{\sqrt{2}}=\frac{3\sqrt{2}}{\sqrt{2}\times\sqrt{2}}=\frac{3\sqrt{2}}{2}.$$

260. It is easy to rationalize the denominator of a fraction when that denominator is a *binomial* involving only quadratic surds. The factor required will consist of the same terms as the given denominator, but with a different sign between them. Thus, $\frac{7-3\sqrt{5}}{6+2\sqrt{5}}$ will have its denominator rationalized by multiplying both terms of the fraction by $6-2\sqrt{5}$. For,

$$\frac{7-3\sqrt{5}}{6+2\sqrt{5}}=\frac{(7-3\sqrt{5})(6-2\sqrt{5})}{(6+2\sqrt{5})(6-2\sqrt{5})}$$

$$=\frac{72-32\sqrt{5}}{16}=\frac{9}{2}-2\sqrt{5}.$$

261. By two operations the denominator of a fraction may be rationalized when that denominator consists of *three* quadratic surds.

Thus, if the denominator be $\sqrt{6}+\sqrt{3}-\sqrt{2}$, both terms of the fraction may be multiplied by $\sqrt{6}-\sqrt{3}+\sqrt{2}$. The resulting denominator will be $6-5+2\sqrt{6}=1+2\sqrt{6}$; and if both terms of the resulting fraction be multiplied by $1-2\sqrt{6}$, the denominator will become $1-24=-23$.

Ex. 98.

Find equivalent fractions with rational denominators, for the following:

1. $\dfrac{3}{\sqrt{7}+\sqrt{5}}$; $\dfrac{7}{2\sqrt{5}-\sqrt{6}}$; $\dfrac{4-\sqrt{2}}{1+\sqrt{2}}$; $\dfrac{6}{5-2\sqrt{6}}$.

2. $\dfrac{a}{\sqrt{b}-\sqrt{c}}$; $\dfrac{a+b}{a-\sqrt{b}}$; $\dfrac{2x-\sqrt{xy}}{\sqrt{xy}-2y}$.

Find the approximate values of:

3. $\dfrac{2}{\sqrt{3}}$; $\dfrac{1}{\sqrt{5}-\sqrt{2}}$; $\dfrac{7\sqrt{5}}{\sqrt{7}+\sqrt{3}}$; $\dfrac{7+2\sqrt{10}}{7-2\sqrt{10}}$.

Imaginary Expressions.

262. All imaginary square roots may be reduced to one form.

$$\sqrt{-x^2}=\sqrt{x^2\times(-1)}=x\sqrt{-1}.$$
$$\sqrt{-a}=\sqrt{a\times(-1)}=a^{\frac{1}{2}}\sqrt{-1}.$$

263. $\sqrt{-1}$ means an expression which, when multiplied by itself, produces -1. Therefore,

$$(\sqrt{-1})^2=-1;$$
$$(\sqrt{-1})^3=(\sqrt{-1})^2\times\sqrt{-1}=-1\sqrt{-1}=-\sqrt{-1};$$
$$(\sqrt{-1})^4=(\sqrt{-1})^2\times(\sqrt{-1})^2=(-1)\times(-1)=1;$$

and so on. So that the successive powers of $\sqrt{-1}$ form the repeating series, $+\sqrt{-1}$, -1, $-\sqrt{-1}$, $+1$.

(1) Multiply $1+\sqrt{-4}$ by $1-\sqrt{-4}$.

$$1+\sqrt{-4}=1+2\sqrt{-1};$$
$$1-\sqrt{-4}=1-2\sqrt{-1}.$$
$$(1+2\sqrt{-1})(1-2\sqrt{-1})=1-4(-1)=5.$$

(2) Divide $\sqrt{-ab}$ by $\sqrt{-b}$.

$$\sqrt{-ab}=a^{\frac{1}{2}}b^{\frac{1}{2}}\sqrt{-1},$$

and $$\sqrt{-b}=b^{\frac{1}{2}}\sqrt{-1}.$$

$$\frac{\sqrt{-ab}}{\sqrt{-b}}=\frac{a^{\frac{1}{2}}b^{\frac{1}{2}}\sqrt{-1}}{b^{\frac{1}{2}}\sqrt{-1}}=\sqrt{a}.$$

Ex. 99.

Multiply:

1. $4+\sqrt{-3}$ by $4-\sqrt{-3}$; $\sqrt{3}-2\sqrt{-2}$ by $\sqrt{3}+2\sqrt{-2}$.
2. $\sqrt{54}$ by $\sqrt{-2}$; $a\sqrt{-b}$ by $x\sqrt{-y}$.
3. $\sqrt{-a}+\sqrt{-b}$ by $\sqrt{-a}-\sqrt{-b}$; $a\sqrt{-a^2b^4}$ by $\sqrt{-a^4b^5}$.
4. $\sqrt{-10}$ by $\sqrt{-2}$; $2\sqrt{3}-6\sqrt{-5}$ by $4\sqrt{3}-\sqrt{-5}$.

Divide:

5. $x\sqrt{-1}$ by $y\sqrt{-1}$; 1 by $\sqrt{-1}$; a by $a^{\frac{1}{2}}\sqrt{-1}$.
6. $\sqrt{-12}$ by $\sqrt{-3}$; $\sqrt{15}$ by $\sqrt{-5}$; $\sqrt{-5}$ by $\sqrt{-20}$.

Equations containing Radicals.

264. An equation containing a *single* radical may be solved by arranging the terms so as to have the radical alone on one side, and then raising both sides to a power corresponding to the order of the radical.

Ex. $\sqrt{x^2-9}+x=9.$

$$\sqrt{x^2-9}=9-x.$$

By squaring, $$x^2-9=81-18x+x^2.$$

$$18x=90.$$

$$\therefore x=5.$$

265. If *two* radicals be involved, two steps may be necessary.

Ex. $\sqrt{x+15}+\sqrt{x}=15.$

$$\sqrt{x+15}+\sqrt{x}=15.$$

By squaring,

$$x+15+2\sqrt{x^2+15x}+x=225.$$

By transposing, $$2\sqrt{x^2+15x}=210-2x.$$

By dividing by 2, $$\sqrt{x^2+15x}=105-x.$$

By squaring, $$x^2+15x=11025-210x+x^2.$$

$$225x=11025.$$

$$\therefore x=49.$$

Some of the following radical equations will reduce to simple and others to quadratic equations.

Ex. 100.

Solve:

1. $\sqrt{x-5}=2.$
2. $2\sqrt{3x+4}-x=4.$
3. $3-\sqrt{x^2-1}=2x.$
4. $\sqrt{3x-2}=2(x-4).$
5. $4x-12\sqrt{x}=16.$
6. $\sqrt{x+4}+\sqrt{2x-1}=6.$
7. $\sqrt{13x-1}-\sqrt{2x-1}=5.$
8. $\sqrt{4+x}+\sqrt{x}=3.$
9. $\sqrt{25+x}+\sqrt{25-x}=8.$
10. $x^2=21+\sqrt{x^2-9}.$
11. $2x-\sqrt[3]{8x^3+26}+2=0.$
12. $\sqrt{x+1}+\sqrt{x+16}=\sqrt{x+25}.$
13. $\sqrt{2x+1}-\sqrt{x+4}=\frac{1}{3}\sqrt{x-3}.$

14. $\sqrt{x+3}+\sqrt{x+8}=5\sqrt{x}$.

15. $\sqrt{3+x}+\sqrt{x}=\dfrac{6}{\sqrt{3+x}}$.

16. $\sqrt{x^2-1}+6=\dfrac{16}{\sqrt{x^2-1}}$.

17. $\dfrac{1}{\sqrt{x+1}}+\dfrac{1}{\sqrt{x-1}}=\dfrac{1}{\sqrt{x^2-1}}$.

18. $\dfrac{\sqrt{x+2a}-\sqrt{x-2a}}{\sqrt{x-2a}+\sqrt{x+2a}}=\dfrac{x}{2a}$.

19. $\sqrt{(x-a)^2+2ab+b^2}=x-a+b$.

20. $\sqrt{(x+a)^2+2ab+b^2}=b-a-x$.

21. $x^3-4x^{\frac{3}{2}}=96$.

22. $x+x^{-1}=2.9$.

23. $x^{\frac{1}{2}}+2a^2x^{-\frac{1}{2}}=3a$.

24. $81\sqrt[3]{x}+\dfrac{81}{\sqrt[3]{x}}=52x$.

266. Equations may be solved with respect to an *expression* in the same manner as with respect to a letter.

(1) Solve $(x^2-x)^2-8(x^2-x)+12=0$.

Consider (x^2-x) as the unknown quantity.

Then $\qquad (x^2-x)^2-8(x^2-x)=-12$.

Complete the square, $(x^2-x)^2-(\quad)+16=4$.

Extract the root, $\qquad (x^2-x)-4=\pm 2$.

$$x^2-x=6 \text{ or } 2.$$

Complete the square, $\qquad 4x^2-(\quad)+1=25 \text{ or } 9$.

Extract the root, $\qquad 2x-1=\pm 5 \text{ or } \pm 3$.

$$2x=6,\ -4,\ 4,\ -2.$$

$$\therefore\ x=3,\ -2,\ 2,\ -1.$$

(2) Solve $5x - 7x^2 - 8\sqrt{7x^2 - 5x + 1} = 8.$

Change the signs, and annex $+1$ to both sides.

$$7x^2 - 5x + 1 + 8\sqrt{7x^2 - 5x + 1} = -7.$$

Solve with respect to $\sqrt{7x^2 - 5x + 1}.$

$$(7x^2 - 5x + 1) + 8(7x^2 - 5x + 1)^{\frac{1}{2}} + 16 = 9.$$

$$(7x^2 - 5x + 1)^{\frac{1}{2}} + 4 = \pm 3.$$

$$(7x^2 - 5x + 1)^{\frac{1}{2}} = -1 \text{ or } -7.$$

Square, $\qquad 7x^2 - 5x + 1 = 1 \text{ or } 49.$

Transpose, $\qquad 7x^2 - 5x = 0 \text{ or } 48.$

From $7x^2 - 5x = 0$, $\qquad x = 0 \text{ or } \frac{5}{7}.$

From $7x^2 - 5x = 48$, $\qquad x = 3 \text{ or } -2\frac{2}{7}.$

NOTE. In verifying the values of x in the original equation, it is seen that the value of $\sqrt{7x^2 - 5x + 1}$ is negative. Thus, by putting 0 for x the equation becomes $0 - 8\sqrt{1} = 8$; and by taking -1 for $\sqrt{1}$ we have $(-8)(-1) = 8$; that is, $8 = 8$.

(3) Solve $x^2 + x + 1 + \dfrac{1}{x} + \dfrac{1}{x^2} = 1.$

Arrange as follows: $\left(x^2 + \dfrac{1}{x^2}\right) + \left(x + \dfrac{1}{x}\right) = 0.$

By adding 2 to $\left(x^2 + \dfrac{1}{x^2}\right)$,

there is obtained $\qquad x^2 + 2 + \dfrac{1}{x^2} = \left(x + \dfrac{1}{x}\right)^2.$

$$\therefore \left(x + \frac{1}{x}\right)^2 + \left(x + \frac{1}{x}\right) = 2.$$

Multiply by 4 and complete the square,

$$4\left(x + \frac{1}{x}\right)^2 + (\ ) + 1 = 9.$$

Extract the root, $\qquad 2\left(x + \dfrac{1}{x}\right) + 1 = \pm 3.$

$$2\left(x + \frac{1}{x}\right) = 2 \text{ or } -4.$$

$$x + \frac{1}{x} = 1 \text{ or } -2.$$

Multiply by x, $x^2 - x = -1$, and $x^2 + 2x = -1$.

$\therefore\ 4x^2 - (\ ) + 1 = -3$, $\therefore\ x^2 + 2x + 1 = 0$.

$2x - 1 = \pm\sqrt{-3}$, $x + 1 = 0$.

$\therefore\ x = \frac{1}{2}(1 \pm \sqrt{-3})$; $\therefore\ x = -1$.

267. An equation like that of (3) which will remain unaltered when $\frac{1}{x}$ is substituted for x, is called a **reciprocal equation.**

It will be found that every reciprocal equation of *odd* degree will be divisible by $x - 1$ or $x + 1$ according as the last term is negative or positive; and every reciprocal equation of *even* degree *with its last term negative* will be divisible by $x^2 - 1$. In every case the equation resulting from the division will be reciprocal.

(4) Solve $x^5 + 2x^4 - 3x^3 - 3x^2 + 2x + 1 = 0$.

This is a reciprocal equation, for, if x^{-1} be put for x, the equation becomes $x^{-5} + 2x^{-4} - 3x^{-3} - 3x^{-2} + 2x^{-1} + 1 = 0$, which multiplied by x^5 gives $1 + 2x - 3x^2 - 3x^3 + 2x^4 + x^5 = 0$, the same as the original equation.

The equation may be written $(x^5 + 1) + 2x(x^3 + 1) - 3x^2(x + 1) = 0$, which is obviously divisible by $x + 1$. The result from dividing by $x + 1$ is $x^4 + x^3 - 4x^2 + x + 1 = 0$, or $(x^4 + 1) + x(x^2 + 1) = 4x^2$. By adding $2x^2$ to $(x^4 + 1)$ it becomes $(x^4 + 2x^2 + 1) = (x^2 + 1)^2$.

Then $(x^2 + 1)^2 + x(x^2 + 1) = 6x^2$.

Multiply by 4 and complete the square,

$$4(x^2 + 1)^2 + (\ ) + x^2 = 25x^2.$$

Extract the root, $2(x^2 + 1) + x = \pm 5x$.

Hence, $2x^2 + 2 = 4x$ or $-6x$.

By simplifying, $x^2 - 2x = -1$; and $x^2 + 3x = -1$,

whence, $x = 1$ and 1; whence, $x = \frac{1}{2}(-3 \pm \sqrt{5})$.

Therefore, including the root -1 obtained from the factor $x + 1$, the *five* roots are $-1, 1, 1, \frac{1}{2}(-3 \pm \sqrt{5})$.

By this process a reciprocal *cubic* equation may be reduced to a *quadratic*, and one of the *fifth* or *sixth* degree to a *biquadratic*, the solution of which may be easily effected.

Ex. 101.

Solve:

1. $x^2 - 3x - 6\sqrt{x^2 - 3x - 3} + 2 = 0.$

2. $x^2 + 3x - \frac{3}{x} + \frac{1}{x^2} = \frac{7}{36}.$

3. $(2x^2 - 3x)^2 - 2(2x^2 - 3x) = 15.$

4. $(ax - b)^2 + 4a(ax - b) = \frac{9a^2}{4}.$

5. $3(2x^2 - x) - (2x^2 - x)^{\frac{1}{2}} = 2.$

6. $15x - 3x^2 + 4(x^2 - 5x + 5)^{\frac{1}{2}} = 16.$

7. $x^2 + x^{-2} + x + x^{-1} = 4.$

8. $x^2 + \sqrt{x^2 - 7} = 19.$

9. $x^2 + x + \frac{1}{6}(x^2 + x)^{\frac{1}{2}} = \frac{7}{6}.$

10. $(x+1)^{\frac{1}{2}} + (x-1)^{\frac{1}{2}} = 5.$

11. $x - 1 = 2 + 2x^{-\frac{1}{2}}.$

12. $\sqrt{3x+5} - \sqrt{3x-5} = 4.$

13. $(x^4 + 1) - x(x^2 + 1) = -2x^2.$

14. $2x^2 - 2\sqrt{2x^2 - 5x} = 5(x + 3).$

15. $x + 2 - 4x\sqrt{x + 2} = 12x^2.$

16. $\sqrt{2x + a} + \sqrt{2x - a} = b.$

17. $\sqrt{9x^2 + 21x + 1} - \sqrt{9x^2 + 6x + 1} = 3x.$

18. $x^{\frac{4}{3}} - 4x^{\frac{2}{3}} + x^{-\frac{4}{3}} + 4x^{-\frac{2}{3}} = -\frac{7}{4}.$

19. $\left.\begin{array}{l} (2x + 3y)^2 - 2(2x + 3y) = 8 \\ x^2 - y^2 = 21 \end{array}\right\}$

20. $\left.\begin{array}{l} x + y + \sqrt{x + y} = a \\ x - y + \sqrt{x - y} = b \end{array}\right\}$

21. $\left.\begin{array}{l} x^4 - x^2y^2 + y^4 = 13 \\ x^2 - xy + y^2 = 3 \end{array}\right\}$

22. $\left.\begin{array}{l} x^2 + y^2 + x + y = 48 \\ xy = 12 \end{array}\right\}$

23. $\left.\begin{array}{l} x^2 + xy + y^2 = a^2 \\ x + \sqrt{xy} + y = b \end{array}\right\}$

CHAPTER XVII.

RATIO AND PROPORTION.

268. The *relative magnitude* of two numbers is called their **ratio**, and is expressed by the fraction which the first is of the second.

Thus, the ratio of 6 to 3 is indicated by the fraction $\frac{6}{3}$, which is sometimes written $6:3$.

269. The first term of a ratio is called the **antecedent**, and the second term the **consequent**. When the antecedent is *equal* to the consequent, the ratio is called a *ratio of equality;* when the antecedent is *greater* than the consequent, the ratio is called a *ratio of greater inequality;* when *less*, a *ratio of less inequality.*

270. When the antecedent and consequent are interchanged, the resulting ratio is called the *inverse* of the given ratio.

Thus, the ratio $3:6$ is the *inverse* of the ratio $6:3$.

271. The ratio of two *quantities* that can be expressed in *integers* in terms of a *common unit* is equal to the ratio of the two *numbers* by which they are expressed.

Thus, the ratio of \$9 to \$11 is equal to the ratio of $9:11$; and the ratio of a line $2\frac{2}{3}$ inches long to a line $3\frac{3}{4}$ inches long, when both are expressed in terms of a unit $\frac{1}{12}$ of an inch long, is equal to the ratio of 32 to 45.

272. Two quantities *different in kind* can have no ratio, for then one cannot be a fraction of the other.

273. Two quantities that can be expressed in integers in terms of a common unit are said to be *commensurable.* The common unit is called a *common measure*, and each quantity is called a *multiple* of this common measure.

Thus, a common measure of $2\frac{1}{2}$ feet and $3\frac{2}{3}$ feet is $\frac{1}{6}$ of a foot, which is contained 15 times in $2\frac{1}{2}$ feet, and 22 times in $3\frac{2}{3}$ feet. Hence, $2\frac{1}{2}$ feet and $3\frac{2}{3}$ feet are multiples of $\frac{1}{6}$ of a foot, $2\frac{1}{2}$ feet being obtained by taking $\frac{1}{6}$ of a foot 15 times, and $3\frac{2}{3}$ by taking $\frac{1}{6}$ of a foot 22 times.

274. When two quantities are *incommensurable*, that is, have no common unit in terms of which *both* quantities can be expressed in *integers*, it is impossible to find a fraction that will indicate the exact value of the ratio of the given quantities. It is possible, however, by taking the unit sufficiently small, to find a fraction that shall differ from the true value of the ratio by as little as we please.

Thus, if a and b denote the diagonal and side of a square,

$$\frac{a}{b} = \sqrt{2}.$$

Now $\sqrt{2} = 1.41421356.....$, a value greater than 1.414213, but less than 1.414214.

If, then, a *millionth part* of b be taken as the unit, the value of the ratio $\frac{a}{b}$ lies between $\frac{1414213}{1000000}$ and $\frac{1414214}{1000000}$, and therefore differs from either of these fractions by less than $\frac{1}{1000000}$.

By carrying the decimal farther, a fraction may be found that will differ from the true value of the ratio by less than a *billionth, trillionth, or any other assigned value whatever.*

275. Expressed generally, when a and b are incommensurable, and b is divided into any integral number (n) of equal parts, if one of these parts be contained in a more than m times, but less than $m+1$ times, then

$$\frac{a}{b} > \frac{m}{n}, \text{ but} < \frac{m+1}{n};$$

that is, the value of $\frac{a}{b}$ lies between $\frac{m}{n}$ and $\frac{m+1}{n}$.

The error, therefore, in taking either of these values for $\frac{a}{b}$ is $< \frac{1}{n}$. But by increasing n indefinitely, $\frac{1}{n}$ can be made to decrease indefinitely, and to become less than any assigned value, however small, though it cannot be made absolutely equal to zero.

276. The ratio between two incommensurable quantities is called an **incommensurable ratio.**

277. *A ratio will not be altered if both its terms be multiplied by the same number.*

For the ratio $a : b$ is represented by $\frac{a}{b}$, the ratio $ma : mb$ is represented by $\frac{ma}{mb}$; and since $\frac{ma}{mb} = \frac{a}{b}$, $\therefore\ ma : mb = a : b$.

278. *A ratio will be altered if different multipliers of its terms be taken; and will be increased or diminished according as the multiplier of the antecedent is greater or less than that of the consequent.*

For, $\quad ma : nb$ will be $>$ or $< a : b$

according as $\quad \frac{ma}{nb}$ is $>$ or $< \frac{a}{b}\left(= \frac{na}{nb}\right)$,

as $\quad ma$ is $>$ or $< na$,

as $\quad m$ is $>$ or $< n$.

279. *A ratio of greater inequality will be diminished, and a ratio of less inequality increased by adding the same number to both its terms.*

For, $\quad a + x : b + x$ is $>$ or $< a : b$

according as $\quad \frac{a+x}{+x}$ is $>$ or $< \frac{a}{b}$,

as $\quad ab + bx$ is $>$ or $< ab + ax$,

as $\quad bx$ is $>$ or $< ax$,

as $\quad b$ is $>$ or $< a$.

280. *A ratio of greater inequality will be increased, and a ratio of less inequality diminished, by subtracting the same number from both its terms.*

For, $a - x : b - x$ will be $>$ or $<$ $a : b$

according as $\dfrac{a-x}{b-x}$ is $>$ or $< \dfrac{a}{b}$,

as $ab - bx$ is $>$ or $< ab - ax$,

as ax is $>$ or $< bx$,

as a is $>$ or $< b$.

281. Ratios are *compounded* by taking the product of the fractions that represent them.

Thus, the ratio compounded of $a : b$ and $c : d$ is found by taking the product of $\frac{a}{b}$ and $\frac{c}{d} = \frac{ac}{bd}$.

The ratio compounded of $a : b$ and $a : b$ is the *duplicate* ratio $a^2 : b^2$, and the ratio compounded of $a : b$, $a : b$, and $a : b$ is the *triplicate* ratio $a^3 : b^3$.

282. Ratios are *compared* by comparing the fractions that represent them.

Thus, $a : b$ is $>$ or $< c : d$

according as $\dfrac{a}{b}$ is $>$ or $< \dfrac{c}{d}$,

as $\dfrac{ad}{bd}$ is $>$ or $< \dfrac{bc}{bd}$,

as ad is $>$ or $< bc$.

Ex. 102.

1. Write down the ratio compounded of $3 : 5$ and $8 : 7$. Which of these ratios is increased and which is diminished by the composition?

2. Compound the duplicate ratio of $4 : 15$ with the triplicate of $5 : 2$.

3. Show that a duplicate ratio is greater or less than its simple ratio according as it is a ratio of greater or less inequality.

4. Arrange in order of magnitude the ratios $3:4$; $23:25$; $10:11$; and $15:16$.

5. Arrange in order of magnitude
$a+b:a-b$ and $a^2+b^2:a^2-b^2$, if $a>b$.

Find the ratios compounded of:

6. $3:5$; $10:21$; $14:15$. 7. $7:9$; $102:105$; $15:17$.

8. Two numbers are in the ratio $2:3$, and if 9 be added to each, they are in the ratio $3:4$. Find the numbers.

(Let $2x$ and $3x$ represent the numbers.)

9. Show that the ratio $a:b$ is the duplicate of the ratio $a+c:b+c$, if $c^2=ab$.

10. Find two numbers in the ratio $3:4$, of which the sum is to the sum of their squares in the ratio of 7 to 50.

11. If five gold coins and four silver ones be worth as much as three gold coins and twelve silver ones, find the ratio of the value of a gold coin to that of a silver one.

12. If eight gold and nine silver coins be worth as much as six gold and nineteen silver coins, find the ratio of the value of a silver coin to that of a gold one.

13. There are two roads from A to B, one of them 14 miles longer than the other; and two roads from B to C, one of them 8 miles longer than the other. The distance from A to B is to the distance from B to C, by the shorter roads, as 1 to 2; by the longer roads, as 2 to 3. Find the distances.

14. A rectangular field contains 5270 acres, and its length is to its breadth in the ratio of $31:17$. Find its dimensions.

283. An equation consisting of two equal ratios is called a **proportion**; and the terms of the ratios are called **proportionals**.

284. The algebraic test of a proportion is that the two fractions which represent the ratios shall be equal.

Thus, the ratio $a:b$ will be equal to the ratio $c:d$ if $\frac{a}{b}=\frac{c}{d}$; and the four *numbers* a, b, c, d are called proportionals, or are said to be in proportion.

285. If the ratios $a:b$ and $c:d$ form a proportion, the proportion is written

$$a:b=c:d$$

(read the ratio of a to b is equal to the ratio of c to d),

or $$a:b::c:d$$

(read a is to b in the same ratio as c is to d).

The first and last terms, a and d, are called the **extremes.**

The two middle terms, b and c, are called the **means.**

286. *When four numbers are in proportion, the product of the extremes is equal to the product of the means.*

For, if $$a:b::c:d,$$

then $$\frac{a}{b}=\frac{c}{d}.$$

By multiplying by bd, $ad=bc$.

287. *If the product of two numbers be equal to the product of two others, either two may be made the extremes of a proportion, and the other two the means.*

For, if $$ad=bc,$$

by dividing by bd, $$\frac{ad}{bd}=\frac{bc}{bd},$$

or $$\frac{a}{b}=\frac{c}{d}.$$

$$\therefore a:b::c:d.$$

288. The equation $ad = bc$ gives

$$a = \frac{bc}{d}; \; b = \frac{ad}{c};$$

so that an extreme may be found by dividing the product of the means by the other extreme; and a mean may be found by dividing the product of the extremes by the other mean.

289. If four quantities, a, b, c, d, be in proportion, they will be in proportion by:

I. **Inversion.**

That is, b will be to a as d is to c.

For, if $$a : b :: c : d,$$

then $$\frac{a}{b} = \frac{c}{d},$$

and $$1 \div \frac{a}{b} = 1 \div \frac{c}{d},$$

or $$\frac{b}{a} = \frac{d}{c}.$$

$$\therefore b : a :: d : c.$$

290. II. **Composition.**

That is, $a + b$ will be to b as $c + d$ is to d.

For, if $$a : b :: c : d,$$

then $$\frac{a}{b} = \frac{c}{d},$$

and $$\frac{a}{b} + 1 = \frac{c}{d} + 1,$$

or $$\frac{a+b}{b} = \frac{c+d}{d}.$$

$$\therefore a + b : b :: c + d : d.$$

291. III. **Division.**

That is, $a - b$ will be to b as $c - d$ is to d.

For, if $$a : b :: c : d,$$

then $$\frac{a}{b} = \frac{c}{d},$$

and $$\frac{a}{b}-1=\frac{c}{d}-1,$$

or $$\frac{a-b}{b}=\frac{c-d}{d}.$$

$$\therefore a-b:b::c-d:d.$$

292. IV. **Composition and Division.**

That is, $a+b$ will be to $a-b$ as $c+d$ is to $c-d$.

For, from II., $$\frac{a+b}{b}=\frac{c+d}{d},$$

and from III., $$\frac{a-b}{b}=\frac{c-d}{d}.$$

By dividing, $$\frac{a+b}{a-b}=\frac{c+d}{c-d}.$$

$$\therefore a+b:a-b::c+d:c-d.$$

293. When the four quantities a, b, c, d are all of the *same kind*, they will be in proportion by:

V. **Alternation.**

That is, a will be to c as b is to d.

For, if $$a:b::c:d,$$

then $$\frac{a}{b}=\frac{c}{d}.$$

By multiplying by $\frac{b}{c}$, $$\frac{ab}{bc}=\frac{bc}{cd},$$

or $$\frac{a}{c}=\frac{b}{d}.$$

$$\therefore a:c::b:d.$$

294. From the proportion $a:c::b:d$ may be obtained by:

VI. **Composition.** $a+c:c::b+d:d.$

VII. **Division.** $a-c:c::b-d:d.$

VIII. **Composition and Division.** $a+c:a-c::b+d:b-d.$

295. *In a series of equal ratios, the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.*

For, if $$\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\frac{g}{h},$$

r may be put for each of these ratios.

Then $$\frac{a}{b}=r,\ \frac{c}{d}=r, \frac{e}{f}=r,\ \frac{g}{h}=r.$$

$$\therefore a=br,\ c=dr,\ e=fr,\ g=hr.$$

$$\therefore a+c+e+g=(b+d+f+h)r.$$

$$\therefore \frac{a+c+e+g}{b+d+f+h}=r=\frac{a}{b}.$$

$$\therefore a+c+e+g:b+d+f+h::a:b.$$

In like manner, it may be shown that

$$ma+nc+pe+qg:mb+nd+pf+qh::a:b.$$

296. If a, b, c, d be in *continued* proportion, that is, if $a:b=b:c=c:d$, then will $a:c=a^2:b^2$ and $a:d=a^3:b^3$.

For, $$\frac{a}{b}=\frac{b}{c}=\frac{c}{d}.$$

Hence, $$\frac{a}{b}\times\frac{b}{c}=\frac{a}{b}\times\frac{a}{b},$$

or $$\frac{a}{c}=\frac{a^2}{b^2}.$$

$$\therefore a:c=a^2:b^2.$$

So $$\frac{a}{b}\times\frac{b}{c}\times\frac{c}{d}=\frac{a}{b}\times\frac{a}{b}\times\frac{a}{b},$$

or $$\frac{a}{d}=\frac{a^3}{b^3}.$$

$$\therefore a:d=a^3:b^3.$$

297. If a, b, c be proportionals, so that $a:b::b:c$, then b is called a *mean* proportional between a and c, and c is called a *third* proportional to a and b.

If $a:b::b:c$, then $b=\sqrt{ac}$.

For, if $a:b::b:c$,

then $\frac{a}{b}=\frac{b}{c}$,

and $b^2=ac$.

$\therefore b=\sqrt{ac}$.

298. *The products of the corresponding terms of two or more proportions are in proportion.*

For, if $a:b::c:d$,

$e:f::g:h$,

and $k:l::m:n$,

then $\frac{a}{b}=\frac{c}{d}$, $\frac{e}{f}=\frac{g}{h}$, $\frac{k}{l}=\frac{m}{n}$.

Hence, by finding the product of the left members, and also of the right members of these equations,

$$\frac{aek}{bfl}=\frac{cgm}{dhn}.$$

$$\therefore aek:bfl::cgm:dhn.$$

299. *Like powers, or like roots, of the terms of a proportion are in proportion.*

For, if $a:b::c:d$,

then $\frac{a}{b}=\frac{c}{d}$

By raising both sides to the *nth* power,

$$\frac{a^n}{b^n}=\frac{c^n}{d^n}.$$

$$\therefore a^n:b^n::c^n:d^n.$$

By extracting the *nth* root,

$$\frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}}=\frac{c^{\frac{1}{n}}}{d^{\frac{1}{n}}}.$$

$$\therefore a^{\frac{1}{n}}:b^{\frac{1}{n}}::c^{\frac{1}{n}}:d^{\frac{1}{n}}.$$

300. *If two quantities be increased or diminished by like parts of each, the results will be in the same ratio as the quantities themselves.*

For,
$$\frac{a}{b}=\frac{\left(1\pm\frac{m}{n}\right)a}{\left(1\pm\frac{m}{n}\right)b},$$

that is,
$$\frac{a}{b}=\frac{a\pm\frac{m}{n}a}{b\pm\frac{m}{n}b}.$$

$$\therefore a:b::a\pm\frac{m}{n}a:b\pm\frac{m}{n}b.$$

301. The laws that have been established for ratios should be remembered when ratios are expressed in their fractional form.

(1) Solve
$$\frac{x^2+x+1}{x^2-x-1}=\frac{x^2-x+2}{x^2+x-2}.$$

By § 392
$$\frac{2x^2}{2(x+1)}=\frac{2x^2}{-2(x-2)},$$

and this equation is satisfied when $x=0$;

or, dividing by $\frac{2x^2}{2}$,
$$\frac{1}{x+1}=\frac{1}{2-x}.$$

$$\therefore x=\tfrac{1}{2}.$$

(2) If $a:b::c:d$, show that
$$a^2+ab:b^2-ab::c^2+cd:d^2-cd.$$

If
$$\frac{a}{b}=\frac{c}{d},$$

then
$$\frac{a+b}{a-b}=\frac{c+d}{c-d}, \qquad \text{§ 292}$$

and
$$\frac{a}{-b}=\frac{c}{-d},$$

$$\therefore \frac{a}{-b}\times\frac{a+b}{a-b}=\frac{c}{-d}\times\frac{c+d}{c-d}; \qquad \text{§ 298}$$

that is,
$$\frac{a^2+ab}{b^2-ab}=\frac{c^2+cd}{d^2-cd},$$

or
$$a^2+ab:b^2-ab::c^2+cd:d^2-cd.$$

Ex. 103.

If $a : b :: c : d$, prove that:

1. $ma : nb :: mc : nd$.
2. $3a + b : b :: 3c + d : d$.
3. $a + 2b : b :: c + 2d : d$.
4. $a^3 : b^3 :: c^3 : d^3$.
5. $a : a + b :: c : c + d$.
6. $a : a - b :: c : c - d$.
7. $ma + nb : ma - nb :: mc + nd : mc - nd$.
8. $2a + 3b : 3a - 4b :: 2c + 3d : 3c - 4d$.
9. $ma^2 + nc^2 : mb^2 + nd^2 :: a^2 : b^2$.

If $a : b :: b : c$, prove that:

10. $a + b : b + c :: a : b$.
11. $a^2 + ab : b^2 + bc :: a : c$.
12. $a : c :: (a + b)^2 : (b + c)^2$.
13. If $\dfrac{x - y}{l} = \dfrac{y - z}{m} = \dfrac{z - x}{n}$, and x, y, z be unequal, then $l + m + n = 0$.
14. Find x when $x + 5 : 2x - 3 :: 5x + 1 : 3x - 3$.
15. Find x when $x + a : 2x - b :: 3x + b : 4x - a$.
16. Find x and y when $x : 27 :: y : 9$, and $x : 27 :: 2 : x - y$.
17. Find x and y when $x + y + 1 : x + y + 2 :: 6 : 7$.
18. Find x when $x^2 - 4x + 2 : x^2 - 2x - 1 :: x^2 - 4x : x^2 - 2x - 2$.
19. A and B trade with different sums. A gains \$200 and B loses \$50, and now A's stock : B's :: $2 : \frac{1}{2}$. But, if A had gained \$100 and B lost \$85, their stocks would have been as $15 : 3\frac{1}{4}$. Find the original stock of each.
20. A line is divided into two parts in the ratio $2 : 3$, and into two parts in the ratio $3 : 4$; the distance between the points of section is 2. Find the length of the line.

CHAPTER XVIII.

SERIES.

302. A succession of numbers which proceed according to some fixed law is called a **series**; and the successive numbers are called the **terms** of the series.

Thus, by executing the indicated division of $\frac{1}{1-x}$, the series $1 + x + x^2 + x^3 + \cdots\cdots$ is obtained, a series that has an *unlimited* number of terms.

303. A series that is continued *indefinitely* is called an **infinite series**; and a series that comes to an end at some particular term is called a **finite series.**

304. When x is < 1, the more terms we take of the infinite series $1 + x + x^2 + x^3 + \cdots\cdots$, obtained by dividing 1 by $1 - x$, the more nearly does their sum *approach* to the value of $\frac{1}{1-x}$.

Thus, if $x = \frac{1}{3}$, then $\frac{1}{1-x} = \frac{1}{1-\frac{1}{3}} = \frac{3}{2}$, and the series becomes $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots\cdots$, a sum which cannot become equal to $\frac{3}{2}$ however great the number of terms taken, but which may be made to differ from $\frac{3}{2}$ by as little as we please by increasing indefinitely the number of terms.

305. But when x is > 1, the more terms we take of the series $1 + x + x^2 + x^3 + \cdots\cdots$, the more does the sum of the series *diverge* from the value of $\frac{1}{1-x}$.

Thus, if $x = 3$, then $\frac{1}{1-x} = \frac{1}{1-3} = -\frac{1}{2}$, and the series becomes $1 + 3 + 9 + 27 + \cdots\cdots$, a sum which *diverges* more and more from $-\frac{1}{2}$,

the more terms we take, and which may be made to increase indefinitely by increasing indefinitely the number of terms taken.

306. A series whose sum as the number of its terms is indefinitely increased approaches some *fixed finite value as a limit* is called a **converging series**; and a series whose sum increases indefinitely as the number of its terms is increased, is called a **diverging series.**

307. When $x = 1$, the division of 1 by $1 - x$, that is, of 1 by 0, has no meaning, according to the *definition of division;* and any attempt to divide by a divisor that is equal to zero leads to absurd results.

Thus,	$8 + 4 = 8 + 4$;	
by transposing,	$8 - 8 = 4 - 4$;	
or, dividing by $4 - 4$,	$2 = 1$;	a manifest absurdity.

308. When $x = 1$ *very nearly*, then the value of $\dfrac{1}{1-x}$ will be *very great*, and the sum of the series $1 + x + x^2 + x^3 + \cdots\cdots$ will become greater and greater the more terms we take. Hence, by making the denominator $1 - x$ approach indefinitely to zero, the value of the fraction $\dfrac{1}{1-x}$ may be made to increase at pleasure.

309. If the symbol $\circ$ be used to denote a quantity that is less than any assignable quantity, and that may be considered to decrease without limit, not, however, becoming 0, and the symbol $\underline{\infty}$ be used to denote a quantity that is greater than any assignable quantity, and that may be considered to increase without limit, not, however, becoming ∞, then

$$\frac{1}{\circ} = \underline{\infty}.$$

In the same sense $\dfrac{a}{\circ} = \underline{\infty}$, where a represents any value that may be assigned.

310. If x in the fraction $\frac{1-x^5}{1-x}$ be equal to 1, the numerator and denominator will each become 0, and the fraction will assume the form $\frac{0}{0}$.

311. If, however, x in this fraction approach to 1 as its limit, then the denominator $1-x$, inasmuch as it has *some* value, even though less than any assignable value, may be used as a divisor, and the result is $1+x+x^2+x^3+x^4$. Hence, it is evident that though both terms of the fraction become smaller and smaller as $1-x$ approaches to 0, still the numerator becomes more and more nearly *five times* the denominator.

It may be remarked that when the symbol $\frac{0}{0}$ is obtained for the value of the unknown quantity in a problem, the meaning is that the problem has no *definite* solution, but that its conditions are satisfied if any value whatever be taken for the required quantity; and if the symbol $\frac{a}{0}$, in which a denotes any assigned value, be obtained for the value of the unknown quantity, the meaning is that the conditions of the problem are impossible.

312. The number of different series is unlimited, but the only kinds of series that will be considered at this stage of the work are Arithmetical, Geometrical, and Harmonical Series.

Arithmetical Series.

313. A series in which the difference between any two adjacent terms is equal to the difference between any other two adjacent terms, is called an **Arithmetical Series** or an **Arithmetical Progression.**

314. The general representative of such a series will be

$$a,\ a+d,\ a+2d,\ a+3d \ldots\ldots,$$

in which a is the first term and d the common difference;

and the series will be *increasing* or *decreasing* according as d is positive or negative.

315. Since each succeeding term of the series is obtained by adding d to the preceding term, the coefficient of d will always be 1 less than the number of the term, so that the

$$nth \text{ term} = a + (n - 1)\, d.$$

If the *nth* term be denoted by l, this equation becomes

$$l = a + (n - 1)\, d. \tag{1}$$

316. The *arithmetical mean* between two numbers is the number which stands between them, and makes with them an arithmetical series.

317. If a and b denote two numbers, and A their arithmetical mean, then, by the definition of an arithmetical series,

$$A - a = b - A.$$

$$\therefore A = \frac{a + b}{2}. \tag{2}$$

318. Sometimes it is required to insert several arithmetical means between two numbers.

If $m =$ the number of means, then $m + 2 = n$, the whole number of terms; and if $m + 2$ be substituted for n in the equation

$$l = a + (n - 1)\, d,$$

the result is $\quad l = a + (m + 1)\, d.$

By transposing a, $\quad l - a = (m + 1)\, d.$

$$\therefore \frac{l - a}{m + 1} = d. \tag{3}$$

Thus, if it be required to insert six means between 3 and 17, the value of d is found to be $\frac{17 - 3}{6 + 1} = 2$; and the series will be 3, 5, 7, 9, 11, 13, 15, 17.

319. If l denote the last term, a the first term, n the number of terms, d the common difference, and s the sum of the terms, it is evident that

$$\begin{array}{rl} s = & a + (a+d) + (a+2d) + \cdots\cdots + (l-d) + l, \text{ or} \\ s = & l + (l-d) + (l-2d) + \cdots\cdots + (a+d) + a \\ \hline \therefore 2s = & (a+l) + (a+l) + (a+l) + \cdots\cdots + (a+l) + (a+l) \\ = & n(a+l). \end{array}$$

$$\therefore s = \frac{n}{2}(a+l). \tag{4}$$

320. From the two equations,

$$l = a + (n-1)d, \tag{1}$$

$$s = \frac{n}{2}(a+l), \tag{2}$$

any one of the quantities a, d, l, n, s may be found when *three* are given.

Ex. Find n when d, l, s are given.

From (1), $a = l - (n-1)d.$

From (2), $a = \frac{2s - ln}{n}.$

Therefore, $l - (n-1)d = \frac{2s - ln}{n},$

$$\therefore ln - dn^2 + dn = 2s - ln,$$

$$\therefore dn^2 - (2l+d)n = -2s,$$

$$\therefore 4d^2n^2 - (\quad) + (2l+d)^2 = (2l+d)^2 - 8ds,$$

$$\therefore 2dn - (2l+d) = \pm\sqrt{(2l+d)^2 - 8ds},$$

$$\therefore n = \frac{2l + d \pm \sqrt{(2l+d)^2 - 8ds}}{2d}.$$

NOTE. The table on the following page contains the results of the general solution of all possible problems in arithmetical series. The student is advised to work these out, both for the results obtained and for the practice gained in solving literal equations in which the unknown quantities are represented by other letters than x, y, z.

No.	Given.	Required.	Results.
1	$a\ d\ n$	l	$l = a + (n-1)d.$
2	$a\ d\ s$		$l = -\frac{1}{2}d \pm \sqrt{[2ds + (a - \frac{1}{2}d)^2]}.$
3	$a\ n\ s$		$l = \frac{2s}{n} - a.$
4	$d\ n\ s$		$l = \frac{s}{n} + \frac{(n-1)d}{2}.$
5	$a\ d\ n$	s	$s = \frac{1}{2}n[2a + (n-1)d].$
6	$a\ d\ l$		$s = \frac{l+a}{2} + \frac{l^2 - a^2}{2d}.$
7	$a\ n\ l$		$s = (l+a)\frac{n}{2}.$
8	$d\ n\ l$		$s = \frac{1}{2}n[2l - (n-1)d].$
9	$d\ n\ l$	a	$a = l - (n-1)d.$
10	$d\ n\ s$		$a = \frac{s}{n} - \frac{(n-1)d}{2}.$
11	$d\ l\ s$		$a = \frac{1}{2}d \pm \sqrt{(l + \frac{1}{2}d)^2 - 2ds}.$
12	$n\ l\ s$		$a = \frac{2s}{n} - l.$
13	$a\ n\ l$	d	$d = \frac{l-a}{n-1}.$
14	$a\ n\ s$		$d = \frac{2(s-an)}{n(n-1)}.$
15	$a\ l\ s$		$d = \frac{l^2 - a^2}{2s - l - a}.$
16	$n\ l\ s$		$d = \frac{2(nl - s)}{n(n-1)}.$
17	$a\ d\ l$	n	$n = \frac{l-a}{d} + 1.$
18	$a\ d\ s$		$n = \frac{d - 2a \pm \sqrt{(2a-d)^2 + 8ds}}{2d}.$
19	$a\ l\ s$		$n = \frac{2s}{l+a}.$
20	$d\ l\ s$		$n = \frac{2l + d \pm \sqrt{(2l+d)^2 - 8ds}}{2d}.$

Ex. 104.

1. Find the thirteenth term of 5, 9, 13.....
   ninth term of $-3, -1, 1$.....
   tenth term of $-2, -5, -8$.....
   eighth term of $a, a+3b, a+6b$.....
   fifteenth term of $1, \frac{6}{7}, \frac{5}{7}$.....
   thirteenth term of $-48, -44, -40$.....

2. The first term of an arithmetical series is 3, the thirteenth term is 55. Find the common difference.

3. Find the arithmetical mean: between 3 and 12; between -5 and 17.

4. Insert three arithmetical means between 1 and 19; and four means between -4 and 17.

5. The first term of a series is 2, and the common difference $\frac{1}{3}$. What term will be 10?

6. The seventh term of a series, whose common difference is 3, is 11. Find the first term.

7. Find the sum of:

$$5+8+11+\cdots\cdots \text{ to ten terms.}$$
$$-4-1+2+\cdots\cdots \text{ to seven terms.}$$
$$a+4a+7a+\cdots\cdots \text{ to } n \text{ terms.}$$
$$\tfrac{2}{3}+\tfrac{7}{15}+\tfrac{4}{15}+\cdots\cdots \text{ to twenty-one terms.}$$
$$1+2\tfrac{2}{3}+4\tfrac{1}{3}+\cdots\cdots \text{ to twenty terms.}$$

8. The sum of six numbers of an arithmetical series is 27, and the first term is 1. Determine the series.

9. How many terms of the series $-5-2+1+$..... must be taken so that their sum may be 63?

10. The first term is 12, and the sum of ten terms is 10. Find the last term.

11. The arithmetical mean between two numbers is 10, and the mean between the double of the first and the triple of the second is 27. Find the numbers.

12. Find the middle term of eleven terms whose sum is 66.

13. The sum of fifteen terms of an arithmetical series is 600, and the common difference is 5. Find the first term.

14. The sum of three numbers in arithmetical progression is 15, and the sum of their squares is 83. Find the numbers.

Let $x - y$, x, $x + y$ represent the numbers.

15. Find three numbers of an arithmetical series whose sum shall be 21, and the sum of the first and second shall be $\frac{3}{4}$ of the sum of the second and third.

16. Find three numbers whose common difference is 1, such that the product of the second and third exceeds that of the first and second by $\frac{1}{2}$.

17. A travels uniformly 20 miles a day; B starts three days later, and travels 8 miles the first day, 12 the second, and so on, in arithmetical progression. In how many days will B overtake A?

18. A number consists of three digits which are in arithmetical progression; and this number divided by the sum of its digits is equal to 26; but if 198 be added to it, the digits in the units' and hundreds' places will be interchanged. Required the number.

19. The sum of the squares of the extremes of four numbers in arithmetical progression is 200, and the sum of the squares of the means is 136. What are the numbers?

20. Suppose that a body falls through a space of $16\frac{1}{12}$ feet in the first second of its fall, and in each succeeding second $32\frac{1}{6}$ more than in the next preceding one. How far will a body fall in 20 seconds?

Geometrical Series.

321. A series is called a **Geometrical Series** or a **Geometrical Progression** when each succeeding term is obtained by multiplying the preceding term by a *constant multiplier*.

322. The general representative of such a series will be

$$a, ar, ar^2, ar^3, ar^4 \ldots\ldots,$$

in which a is the first term and r the constant multiplier or ratio.

323. Since the exponent of r increases by 1 for every term, the exponent will always be 1 less than the number of the term; so that the

$$nth \text{ term} = ar^{n-1}.$$

324. If the *nth* term be denoted by l, this equation becomes

$$l = ar^{n-1}. \qquad (1)$$

325. The *geometrical mean* between two numbers is the number which stands between them, and makes with them a geometrical series.

326. If a and b denote two numbers, and G their geometrical mean, then, by definition of a geometrical series,

$$\frac{G}{a} = \frac{b}{G}.$$

$$\therefore G = \sqrt{ab}. \qquad (2)$$

327. Sometimes it is required to insert several geometrical means between two numbers.

If $m=$ the number of means, then $m+2=n$, the whole number of terms; and if $m+2$ be substituted for n in the equation

$$l=ar^{n-1},$$

the result is

$$l=ar^{m+1}.$$

$$\therefore\ r^{m+1}=\frac{l}{a}. \tag{3}$$

Thus, if it be required to insert three geometrical means between 3 and 48, the value of r is found to be

$$r^4=\frac{48}{3}=16.$$

$$\therefore\ r=2,$$

and the series will be $\quad 3, 6, 12, 24, 48.$

328. If l denote the last term, a the first term, n the number of terms, r the common ratio, and s the sum of the n terms, then

$$s=a+ar+ar^2+ar^3+\cdots\cdots+ar^{n-1}.$$

Multiply by r, $rs=ar+ar^2+ar^3+\cdots\cdots+ar^{n-1}+ar^n.$

Therefore, by subtracting the first equation from the second,

$$rs-s=ar^n-a,$$

or

$$(r-1)\,s=a\,(r^n-1).$$

$$\therefore\ s=\frac{a\,(r^n-1)}{r-1}. \tag{4}$$

329. When r is <1, this formula will be more convenient if written

$$s=\frac{a\,(1-r^n)}{1-r}.$$

330. Since

$$l=ar^{n-1},$$

$$rl=ar^n,$$

and (4) may be written $s=\dfrac{rl-a}{r-1}.$

In working out the following results, the student will make use of the two equations, $l=ar^{n-1}$ and $s=\dfrac{a\,(r^n-1)}{r-1}.$

No.	Given.	Required.	Results.
1	$a\ r\ n$	l	$l = ar^{n-1}.$
2	$a\ r\ s$		$l = \dfrac{a + (r-1)s}{r}.$
3	$a\ n\ s$		$l(s-l)^{n-1} - a(s-a)^{n-1} = 0.$
4	$r\ n\ s$		$l = \dfrac{(r-1)sr^{n-1}}{r^n - 1}.$
5	$a\ r\ n$	s	$s = \dfrac{a(r^n - 1)}{r-1}.$
6	$a\ r\ l$		$s = \dfrac{rl - a}{r-1}.$
7	$a\ n\ l$		$s = \dfrac{\sqrt[n-1]{l^n} - \sqrt[n-1]{a^n}}{\sqrt[n-1]{l} - \sqrt[n-1]{a}}.$
8	$r\ n\ l$		$s = \dfrac{lr^n - l}{r^n - r^{n-1}}.$
9	$r\ n\ l$	a	$a = \dfrac{l}{r^{n-1}}.$
10	$r\ n\ s$		$a = \dfrac{(r-1)s}{r^n - 1}.$
11	$r\ l\ s$		$a = rl - (r-1)s.$
12	$n\ l\ s$		$a(s-a)^{n-1} - l(s-l)^{n-1} = 0.$
13	$a\ n\ l$	r	$r = \sqrt[n-1]{\dfrac{l}{a}}.$
14	$a\ n\ s$		$r^n - \dfrac{s}{a}r + \dfrac{s-a}{a} = 0.$
15	$a\ l\ s$		$r = \dfrac{s-a}{s-l}.$
16	$n\ l\ s$		$r^n - \dfrac{s}{s-l}r^{n-1} + \dfrac{l}{s-l} = 0.$
17	$a\ r\ l$	n	$n = \dfrac{\log l - \log a}{\log r} + 1.$
18	$a\ r\ s$		$n = \dfrac{\log[a + (r-1)s] - \log a}{\log r}.$
19	$a\ l\ s$		$n = \dfrac{\log l - \log a}{\log(s-a) - \log(s-l)} + 1.$
20	$r\ l\ s$		$n = \dfrac{\log l - \log[lr - (r-1)s]}{\log r} + 1.$

Ex. 105.

1. Find the seventh term of 2, 6, 18.....
   sixth term of 3, 6, 12.....
   ninth term of 6, 3, $1\frac{1}{2}$.....
   eighth term of 1, -2, 4.....
   twelfth term of x^3, x^4, x^5.....
   fifth term of $4a$, $-6ma^2$, $9m^2a^3$.....

2. Find the geometrical mean between $18x^3y$ and $30xy^3z$.

3. Find the ratio when the first and third terms are 5 and 80 respectively.

4. Insert two geometrical means between 8 and 125; and three between 14 and 224.

5. If $a=2$ and $r=3$, which term will be equal to 162?

6. The fifth term of a geometrical series is 48, and the ratio 2. Find the first and seventh terms.

7. Find the sum of:
   $3+6+12+\cdots\cdots$ to eight terms.
   $1-3+9-\cdots\cdots$ to seven terms.
   $8+4+2+\cdots\cdots$ to ten terms.
   $.1+.5+2.5+\cdots\cdots$ to seven terms.
   $m-\frac{m}{4}+\frac{m}{16}-\cdots\cdots$ to five terms.

8. The population of a city increases in four years from 10,000 to 14,641. What is the rate of increase?

9. The sum of four numbers in geometrical progression is 200, and the first term is 5. Find the ratio.

10. Find the sum of eight terms of a series whose last term is 1, and fifth term $\frac{1}{8}$.

11. In an odd number of terms, show that the product of the first and last will be equal to the square of the middle term.

12. If from a line one-third be cut off, then one-third of the remainder, and so on, what fraction of the whole will remain when this has been done five times?

13. Of three numbers in geometrical progression, the sum of the first and second exceeds the third by 3, and the sum of the first and third exceeds the second by 21. What are the numbers?

14. Find two numbers whose sum is $3\frac{1}{4}$ and geometrical mean $1\frac{1}{2}$.

15. Find three numbers in geometrical progression whose sum is 13 and the sum of their squares 91.

16. The difference between two numbers is 48, and the arithmetical mean exceeds the geometrical by 18. Find the numbers.

17. There are four numbers in geometrical progression, the second of which is less than the fourth by 24, and the sum of the extremes is to the sum of the means as 7 to 3. Find the numbers.

18. A cask contains 240 gallons of wine. The first day of January, half is drawn out; the following day, half of the rest; the third day, half of what then remains; and so on. How much will be drawn off on January 31?

19. The sum of the two extreme terms of a geometrical progression of four terms is 195; the sum of the two mean terms is 60. What is the first term, and what is the ratio?

331. When $r < 1$, a geometrical series has its terms continually decreasing; and by increasing n, the value of the nth term, ar^{n-1} may be made as small as we please, though not absolutely zero.

332. The formula for the sum of n terms,

$$\frac{a(1-r^n)}{1-r}$$

may be written
$$\frac{a}{1-r}-\frac{ar^n}{1-r}.$$

By increasing n indefinitely, the value of $\frac{ar^n}{1-r}$ becomes indefinitely small, so that the sum of n terms approaches indefinitely to $\frac{a}{1-r}$ as its limit.

(1) Find the limit of $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}\cdots\cdots$

Here $a=1$, and $r=-\frac{1}{2}$,

and therefore the limit $\frac{a}{1-r}=\frac{1}{1-(-\frac{1}{2})}=\frac{1}{1+\frac{1}{2}}=\frac{2}{3}.$ *Ans.*

(2) Find the limit of 0.5515151.

The terms after the first term form an infinite series in which $a=0.051$, and $r=0.01$. Hence, the value is

$$\frac{5}{10}+\frac{0.051}{1-0.01}=\frac{5}{10}+\frac{51}{990}=\frac{5}{10}+\frac{17}{330}=\frac{91}{165}.$$

20. Find the limits of the sums of the following infinite series:

$4+2+1+\cdots\cdots$	$2-1\frac{1}{3}+\frac{8}{9}-\cdots\cdots$
$\frac{1}{2}+\frac{1}{3}+\frac{2}{9}+\cdots\cdots$	$.1+.01+.001+\cdots\cdots$
$\frac{1}{4}-\frac{1}{16}+\frac{1}{64}-\cdots\cdots$	$.868686\ldots\ldots$
$1-\frac{2}{5}+\frac{4}{25}-\cdots\cdots$	$.54444\ldots\ldots$
$\frac{1}{5}+\frac{1}{15}+\frac{1}{45}+\cdots\cdots$	$.83636\ldots\ldots$

CHAPTER XIX.

BINOMIAL THEOREM.

333. The Binomial Theorem is a formula by means of which a binomial may be raised to any required power without going through the process of multiplication.

334. The general formula is as follows:

$$(x+a)^n = x^n + nax^{n-1} + \frac{n(n-1)}{1\times 2}a^2x^{n-2} + \frac{n(n-1)(n-2)}{1\times 2\times 3}a^3x^{n-3} + \dots\dots + a^n,$$

in which the laws stated in § 83 for exponents and coefficients hold good.

335. The expression on the right side is called the *expansion* of $(x+a)^n$.

If a and x be interchanged, the expansion will proceed by ascending powers of x, as follows:

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{1\times 2}a^{n-2}x^2 + \dots\dots + nax^{n-1} + x^n.$$

If $a=1$, then

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1\times 2}x^2 + \dots\dots + nx^{n-1} + x^n.$$

If x be negative, the *odd* powers of x will be *negative* and the *even* powers *positive*.

$$(a-x)^n = a^n - na^{n-1}x + \frac{n(n-1)}{1\times 2}a^{n-2}x^2 - \frac{n(n-1)(n-2)}{1\times 2\times 3}a^{n-3}x^3 + \dots\dots$$

336. It will be observed that the *last factor in the denominator* of the coefficient is 1 less than the *number of the term*, and is the same as the *exponent of the second letter;* also, that the *last factor of the numerator* of the coefficient is found by subtracting the last factor in the denominator from $n + 1$, and that the *exponent of the first letter* is found by subtracting the exponent of the second letter from n. So that,

The *rth* (or general) term in the expansion of $(a + x)^n$ is

$$\frac{n(n-1)\ldots\ldots(n-r+2)}{1\times 2\ldots\ldots(r-1)}a^{n-r+1}x^{r-1}.$$

Thus, the *third* term of $(a + x)^{20}$ is

$$\frac{20\times 19}{1\times 2}a^{18}x^2 = 190\,a^{18}x^2.$$

The **Form** of the expansion is the same whether the exponent n is a positive integral number or negative or fractional; but the series will *terminate* only when n is a *positive integral number*.

The proof of the Binomial Theorem is not given here, as it is too difficult for the student at this stage of his progress.

Expand to four terms by substituting in the general formula:

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(-1)}{1\times 2}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{1\times 2\times 3}a^{n-3}x^3.$$

(1) $$(1+x)^{\frac{1}{3}} = 1 + \tfrac{1}{3}x + \frac{\frac{1}{3}(\frac{1}{3}-1)}{1\times 2}x^2 + \frac{\frac{1}{3}(\frac{1}{3}-1)(\frac{1}{3}-2)}{1\times 2\times 3}x^3 + \ldots\ldots = 1 + \tfrac{1}{3}x - \tfrac{1}{9}x^2 + \tfrac{5}{81}x^3 - \ldots\ldots.$$

(2) $$\left(1-\frac{2x}{a}\right)^{-\frac{1}{4}}=1+\frac{x}{2a}+\frac{-\frac{1}{4}(-\frac{1}{4}-1)}{1\times 2}\left(\frac{2x}{a}\right)^2$$
$$-\frac{-\frac{1}{4}(-\frac{1}{4}-1)(-\frac{1}{4}-2)}{1\times 2\times 3}\left(\frac{2x}{a}\right)^3$$
$$=1+\frac{x}{2a}+\frac{5x^2}{8a^2}+\frac{15x^3}{16a^3}+\cdots\cdots.$$

(3) Find the eighth term of

$$(3x^{\frac{1}{2}}-y^{-1})^{11}.$$

In this case $r=8$ and $n=1$, and the eighth term is

$$\frac{11\times 10\times 9\times 8\times 7\times 6\times 5}{1\times 2\times 3\times 4\times 5\times 6\times 7}(3x^{\frac{1}{2}})^4(-y^{-1})^7$$
$$=330(81x^2)(-y^{-7})$$
$$=-26730x^2y^{-7}.$$

A root may often be extracted by means of an expansion.

(4) Extract the cube root of 344 to six decimal places.

$$344=343(1+\tfrac{1}{343})=7^3(1+\tfrac{1}{343}).$$
$$\therefore \sqrt[3]{344}=7(1+\tfrac{1}{343})^{\frac{1}{3}}$$
$$=7\left(1+\tfrac{1}{3}\times\tfrac{1}{343}+\frac{\frac{1}{3}(\frac{1}{3}-1)}{1\times 2}(\tfrac{1}{343})^2+\cdots\cdots\right)$$
$$=7(1+.000971815-.000000944)$$
$$=7.006796.$$

Ex. 106.

Apply the formula to the development of the following expressions:

1. $(\frac{1}{2}p^5+3y^4)^5$.
2. $(\frac{2}{3}a^3+\frac{3}{4}b^2)^5$.
3. $(a^2b^3+2a^3bx^4)^5$.
4. $(\frac{2}{3}ab^2-\frac{3}{4}a^2y)^7$.
5. $(\sqrt{a}+x)^7$.
6. $(\sqrt{2b}-m)^8$.

7. $(\sqrt{3c}+2a)^7$.

8. $(\sqrt{\tfrac{1}{2}a}-3y)^8$.

9. $(2\sqrt{e}+a)^6$.

10. $(\tfrac{2}{3}a+\sqrt{2x})^6$.

11. $(\tfrac{3}{4}a-\sqrt{\tfrac{1}{2}x})^5$.

12. $(2a-3\sqrt{y})^6$.

13. $(a^2+\tfrac{1}{2}\sqrt{z})^7$.

14. $(\sqrt{b}-\sqrt{y})^8$.

15. $(\sqrt{2c}+\sqrt{3x})^6$.

Find, in the following developments, the term demanded:

16. The seventh term of $(2+a)^{16}$.

17. The eleventh term of $(a+d)^{21}$.

18. The sixth term of $(3+2x^2)^9$.

19. The fourteenth term of $(y^3-1)^{40}$.

20. The seventh term of $(\tfrac{1}{2}a-\sqrt{x^3})^{17}$.

21. The fourth term of $(\sqrt{a}-\sqrt[3]{x^2})^{11}$.

22. Develop $(1+x)^{-1}$, and then make $x=\tfrac{1}{9}$.

23. Develop $(1+x)^{\frac{1}{2}}$, and then make $x=\tfrac{3}{8}$.

24. Develop $(1+x)^{-\frac{1}{2}}$, and then make $x=0.003$.

Expand to four terms:

25. $(a+x)^{-5}$.

26. $(a-x)^{-6}$.

27. $(2b-y)^{-6}$.

28. $(\tfrac{1}{2}c+z)^{-9}$.

29. $(a+\tfrac{1}{2}x)^{-9}$.

30. $(a^2+x^3)^{-5}$.

31. $(\sqrt{a}-x^2)^{-6}$.

32. $(b+h)^{\frac{1}{2}}$.

33. $(b-x)^{\frac{1}{4}}$.

34. $(x^2+a)^{\frac{1}{2}}$.

35. $(a^2-1)^{\frac{1}{2}}$.

36. $(1+a)^{\frac{1}{9}}$.

37. $(a^3+1)^{-\frac{1}{2}}$.

38. $(x^2-a)^{-\frac{1}{2}}$.

39. $(1-x^5)^{-\frac{1}{3}}$.

40. $(1+2d)^{-\frac{1}{5}}$.

41. $(32+5h)^{-\frac{3}{5}}$.

42. $(9-2x^2)^{-\frac{3}{2}}$.

Apply the formula of the binomial to the extraction of the following roots, carrying out the operation to the sixth decimal:

43. $\sqrt{53}$. **45.** $\sqrt[4]{68}$. **47.** $\sqrt[5]{1121}$.

44. $\sqrt[3]{87}$. **46.** $\sqrt[5]{259}$. **48.** $\sqrt[3]{58\frac{1}{8}}$.

SHORTER COURSE IN ALGEBRA.

ANSWERS.

Ex. 1.

1. 22.	6. 94.	11. 120.	15. $64\frac{4}{5}$.	19. 0.
2. 26.	7. 81.	12. 25.	16. 31.	20. 40.
3. 564.	8. 16.	13. 43.	17. -15.	21. 0.
4. 6.	9. $1\frac{2}{3}$.	14. 25.	18. 24.	22. $10\frac{1}{2}$.
5. 39.	10. 1.			

Ex. 2.

1. $a+b$. 2. $2x$. 3. $a-5$. 4. $x+1$.

5. $x-2,\ x-1,\ x,\ x+1,\ x+2$.

6. ax. 8. $a-b$. 10. $m+n$.

7. y. 9. $ab+c$. 11. $\frac{ab}{9h}$.

12. $\frac{x+y}{c}=abm-bc+\frac{a}{x+y}$. 13. $\frac{6n^2}{m-a}+5b(c+d-a)$.

Ex. 3.

3. 31. 4. 30, 5. 5. 25 cents.

6. 60 ft. broken off; 15 ft. standing. 7. 27, 36.

8. Horse, \$168; cow, \$42; sheep, \$6.

9. Apples, 13 cents; pears, 26 cents; oranges, 52 cents

10. Harness, \$50; wagon, \$100; horse, \$200.

11. Thomas, \$1.20; Richard, \$1.20; Henry, 60 cents.

12. A, \$100; B, \$400 C, \$500. 13. Boy, 12; Sister, 4 years.

Ex. 4.

1. 5.	3. -11.	5. 51.	7. \$2484.
2. -40.	4. -3.	6. 495.	8. B.C. 241.

9. A.D. 14. 10. 150 steps in all; 76 steps forward.

Ex. 5.

1. 0.
2. $6mx$.
3. $-20mng$.
4. $3x^2$.
5. $7my^2$.
6. $2ab$.
7. $25my$.
8. $-22a$
9. $-55xy$.
10. $14a^2x^2$.
11. $6b^2m^3$.
12. $9a-10b$.
13. $-a^2c+xyz$.
14. $3x^2y-4ab-2mn$.

Ex. 6.

1. $9a+9b+9c$.
2. $a-b$.
3. $2a+2b$.
4. $a+b+c$.
5. $-2a+2b+2d$.
6. $2x^3-9x^2-8x+10$.
7. $5x^4+4x^3+3x^2+2x-9$.
8. $4a^3-4ab^2+2a^2b-7b^2+b^3$.
9. $6ab+7ax^2-9a^2x+ax^3$.
10. $-c^4+c+8$.
11. $-2xy-2y^2+11yz+5z^2-6y-6z$.
12. $6m^5-m^4n+2m^3n^2+3m^2n^3-4mn^4+5n^5$.

Ex. 7.

1. 9.
2. -25.
3. -89.
4. 200.
5. 506 years.
6. 35° in latitude; 75° in longitude.

Ex. 8

1. $9x$.
2. $-8ab$.
3. $-7ab^2$.
4. $22m^2x^2$.
5. $-4ay$.
6. $41ax^3$.
7. $8a^2x$.
8. xy.
9. $8ax+3ay$.
10. $2ab^2y-aby$.
11. $6x^2$.
12. $13x^2y$.
13. $-6ax^3$.
14. $-3ab+6mx$.
15. $3a-2b+4c$.

Ex. 9.

1. $4a+2c$.
2. $a+6b+4c$.
3. $2x^2-2x-4$.
4. $3x^4-x^3-14x+18$.
5. $x^2-ax+2a^2$.
6. $-5xy-5xz+yz+2y^2$.
7. $2a^3-6a^2b+6ab^2-2b^3$.
8. $-4xy+2xz-y^2+5yz-z^2$.
9. $ax^2+7abx-bx^2+b^2x+12b^3+x^3$.
10. $-2x^3+3xy^2-y^3-14x^2+2xy-10y^2+6$.
11. $-a^4-8ab^3+b^4$.
12. $-2x^5+4x^4y-8xy^4-8y^5$.
13. $a^2b^2-3a^2bc-3ab^2c-abc^2-a^2c^2-b^2c^2$.
14. $2a+4b-9c+d$.
15. $b-a$; $a^3+a^2b+6ab^2+b^3$.
16. $x^2-4xy+2z^2$.
17. $19ac+17cd-17$.
18. $-10a^2-4ab+c^2$.
19. $xy-2x-6y+1$.
20. $-2a^2bc-2ab^2c+2abc^2-2abc$.
21. $5x^2-5x+5$.
22. x^2+xy+y^2.
23. $ax^2-by^2-cx^2+dy^2$.
24. $2bx+2by$.
25. $7x-7y+2y^2$.

26. $a^2b^2 + 12abc - 9ax^2 - 4ab^2 + 6acx - 3a^2x$.

27. $-a^2 - 6b^2 + c^2$.

28. $2a^2$.

29. $x^2 - 3y^2 - 3z^2$.

30. $2a^3 - 2a^2c + 2ac^2$.

Ex. 10.

1. $2b$.
2. $a + b - 2c$.
3. $3x - 3y - z$.
4. $a - b + c$.
5. $-2x$.
6. $-9b + 14c$.
7. $a + a^3$.
8. $2a + 3c$.
9. $3a - 3b$.
10. $2a - b$.
11. $5a$.
12. $4x - y - z$.
13. $-b + 10c$.
14. $-5a$.
15. $4a - 16b - 2c$.
16. $3a - 2c$.
17. $16 - 12x$.
18. $4c$.
19. $3a$.
20. a.

Ex. 11.

1. $(2a-3b)-(4c-d)+(3e-2f)$; $(2a-3b-4c)+(d+3e-2f)$.
2. $(a-2x)+(4y-3z)-(2b-c)$; $(a-2x+4y)-(3z+2b-c)$.
3. $(a^5+3a^4)-(2a^3+4a^2)+(a-1)$; $(a^5+3a^4-2a^3)-(4a^2-a+1)$.
4. $-(3a+2b)+(2c-5d)-(e+2f)$; $-(3a+2b-2c)-(5d+e+2f)$.
5. $(ax - by) - (cz + bx) + (cy + az)$; $(ax - by - cz) - (bx - cy - az)$.
6. $(2x^5 - 3x^4y) + (4x^3y^2 - 5x^2y^3) + (xy^4 - 2y^5)$; $(2x^5 - 3x^4y + 4x^3y^2) - (5x^2y^3 - xy^4 + 2y^5)$.
7. $\begin{cases} 1.\ (2a - 3b - 4c) + (d + [3e - 2f]). \\ 2.\ (a - 2x + 4y) - (3z + [2b - c]). \\ 3.\ (a^5 + 3a^4 - 2a^3) - (4a^2 - [a - 1]). \\ 4.\ -(3a + 2b - 2c) - (5d + [e + 2f]). \\ 5.\ (ax - by - cz) - (bx - [cy + az]). \\ 6.\ (2x^5 - 3x^4y + 4x^3y^2) - (5x^2y^3 - [xy^4 - 2y^5]). \end{cases}$
8. $(2a - 4b - 2c)x - (6a + 3c)y + (4b)z$.
9. $(a - b)x + (2a + 3)y + (4a - 3b - 2)z$.
10. $(a - 4b - 2c)x - (2b + 3c + a)y + (a + 4b + 5c)z$.
11. $(12a - 15c)x + (12a + 4b + 6c)y - (12b - 3c)z$.
12. $(2a - 2b)x - (3b + c)y$.

Ex. 12.

1. -136.
2. -320.
3. -18.0621.
4. 90.
5. -258.
6. 45,700.
7. 61,225.
8. 82.477956.
9. -728.
10. 2052.
11. 183.92.
12. -1718.382762.
13. 8.0001703416.

Ex. 13.

9. $-12a^2$.
10. $45m^2n$.
11. $-12abxy$.
12. $-8cdmn$.
13. $-14a^2bc$.
14. $15m^3x^3$.
15. $-10a^{m+n}$.
16. $21a^5x^6$.
17. $224abc$.

18. $-96a^2b^2c^3$. 21. $-42m^4qx^3$. 24. $162x^8y^4z^6$.
19. $-18{,}954a^2bmp^2$. 22. $-144p^6q^6$. 25. $-24a^2b^2m^2x^2$.
20. $-60a^3b^5y^7$. 23. $24a^6m^9x^8$. 26. $-168a^4b^6m^3n^6$.

Ex. 14.

5. $12a^3b-9ab^2$.
6. $24a^4-27a^3b$.
7. $6x^4y-8x^2y^3+10x^3yz^2$.
8. $a^4x^3y-5a^3x^4y+a^2x^5y+2ax^6y$.
9. $27a^6b^4-9a^4b^6+12a^3b^7+3ab^9$.
10. $-15x^5y+10x^4y^2+35x^3y^3-5x^2y^4$.
11. $12x^3y^3-15x^4y^2-24x^5y$.
12. $3a^4-2a^5b-a^6b^2$.
13. $3x^3yz^2+6x^4yz^3-15x^5y^2z^3+18x^6y^3z-9x^6y^3z^2$.

Ex. 15.

1. x^4+x^2-20. 4. x^3-y^3. 7. x^4-2x+1.
2. $y^2+7y-78$. 5. $2x^2+3xy-2y^2$. 8. x^3-9a^2x.
3. a^6-x^6. 6. $6x^4-96$. 9. $-10b^3-ab^2+26a^2b-7a^3$.
10. $2a^2+5ab+2b^2$. 13. $6a^3b-23a^2b^2+20ab^3$.
11. a^3-b^3. 14. $-4a^5-4a^2b^3+16a^3b^2-8ab^4$.
12. a^3+b^3. 15. $a^4+a^2b^2+b^4$.
16. $a^5-5a^4b+10a^3b^2-10a^2b^3+5ab^4-b^5$.
17. $x^2-4y^2+12yz-9z^2$.
18. $6x^4+x^3y-16xy^3+2x^2yz+3xy^2z+4y^3z$.
19. $x^4+x^3y+x^2y^2+x^3z+x^2yz+xy^2z+x^2z^2+xyz^2+y^2z^2$.
20. $a^3-3abc+b^3+c^3$. 22. $x^5+151x-264$.
21. $x^3+3xy+y^3-1$. 23. $x^5-41x-120$.
24. $x^6+10x-33$. 25. $x^7-7x^6+21x^5-17x^4-25x^3+4x^2+2x$.
26. $4x^6-5x^5+8x^4-10x^3-8x^2-5x-4$.
27. $25a^7b-9a^5b^3+22a^4b^4-4a^3b^5+ab^7$.
28. $4a^{13}y^3+8a^{11}y^4-64a^5y^7-128a^3y^8$.
29. $6m^7n-18m^5n^3+18m^3n^5-6mn^7$.
30. $24a^9b-26a^5b^5+4a^4b^6-6a^3b^7-5a^2b^8+4ab^9-3b^{10}$.
31. x^4-10x^2+9. 32. x^8+x^4+1. 33. $a^8+a^4b^4+b^8$.
34. $32a^7b-8a^6b^2+8a^5b^3-6a^4b^4-2a^3b^5-a^2b^6+ab^7$.
35. $x^5+ax^4-27a^2x^3-13a^3x^2+134a^4x+120a^5$.
36. $531{,}441a^{12}-b^{12}$. 37. $-y^2z^2-3z^4$.
38. $3a^4b-a^3b^2-3a^2b^3-6a^2b^2+2ab^3-2ab^4+6b^4-6b^5$.
39. $12y^2-28xy$. 41. $37b^2-ab-2b+2a-9a^2$.
40. b^2-d^2. 42. $2a^2+2b^2+2c^2$.
43. $ac-c^2$. 45. 0. 47. $24ab-120b^2$.
44. $2mn$. 46. $6x$. 48. $2(x^2+y^2-z^2)$.
49. $-2(10a^4-15a^3+7a^2b+12ab-12b^2)$.
50. $3a^2b-2ab^2$. 51. $12y^2-28xy$. 52. 0.

Ex. 16.

7. $x^2+2xy+y^2$.
8. $y^2-2yz+z^2$.
9. $4x^2+4x+1$.
10. $4a^2+20ab+25b^2$.
11. $1-2x^2+x^4$.
12. $9a^2x^2-24ax^3+16x^4$.
13. $1-14a+49a^2$.
14. $25x^2y^2+20xy+4$.
15. $a^2b^2+2abcd+c^2d^2$.
16. $9m^2n^2-24mn+16$.
17. $144+120x+25x^2$.
18. $16x^2y^4-8xy^3z^2+y^2z^4$.
19. $9a^2b^2c^2-6ab^2c^2d+b^2c^2d^2$.
20. $16x^6-8x^4y^2+x^2y^4$.
21. x^2-y^2.
22. $4a^2-b^2$.
23. $9-x^2$.
24. $9a^2b^2-4b^4$.
25. $16x^4-9y^4$.
26. $a^6x^4-b^2y^8$.
27. $36x^2y^2-25y^4$.
28. $16x^{10}-1$.
29. $1-9a^2b^6$.
30. $a^4x^4-b^4y^4$.

Ex. 17.

1. $x^2+y^2+z^2+2xy+2xz+2yz$.
2. $x^2+y^2+z^2-2xy+2xz-2yz$.
3. $m^2+n^2+p^2+q^2+2mn-2mp-2mq-2np-2nq+2pq$.
4. $x^4+4x^2+9+4x^3-6x^2-12x$.
5. $x^4+y^4+z^4+2x^2y^2-2x^2z^2-2y^2z^2$.
6. $x^8+18x^4y^4+y^8-8x^6y^2-8x^2y^6$.
7. $a^6+b^6+c^6+2a^3b^3+2a^3c^3+2b^3c^3$.
8. $x^6+y^6+z^6-2x^3y^3-2x^3z^3+2y^3z^3$.
9. $x^2+4y^2+9z^2+4xy-6xz-12yz$.
10. x^4+4+4x^3-8x.

Ex. 18.

1. x^2+5x+6.
2. x^2+6x+5.
3. $x^2-9x+18$.
4. x^2-9x+8.
5. x^2-7x-8.
6. $x^2+3x-10$.
7. $x^2+4x-21$.
8. x^2-6x+8.
9. $x^2+12x+11$.
10. $x^2+ax-6a^2$.
11. $x^2-(c+d)x+cd$.
12. $x^2-3xy-4y^2$.
13. $a^2-7ba+10b^2$.
14. $x^4+3x^2y^2+2y^4$.

Ex. 19.

1. $x^3+3x^2a+3xa^2+a^3$.
2. $x^3-3x^2a+3xa^2-a^3$.
3. x^3+3x^2+3x+1.
4. x^3-3x^2+3x-1.
5. $x^4+4x^3a+6x^2a^2+4xa^3+a^4$.
6. $x^4-4x^3a+6x^2a^2-4xa^3+a^4$.
7. $x^4+4x^3+6x^2+4x+1$.
8. $x^4-4x^3+6x^2-4x+1$.
9. $x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5$.
10. $x^5-5x^4y+10x^3y^2-10x^2y^3+5xy^4-y^5$.
11. $x^5+5x^4+10x^3+10x^2+5x+1$.
12. $x^5-5x^4+10x^3-10x^2+5x-1$.
13. $b^4+4b^3x+6b^2x^2+4bx^3+x^4$.

14. $d^5+5d^4y+10d^3y^2+10d^2y^3+5dy^4+y^5.$

15. $c^6+6c^5z+15c^4z^2+20c^3z^3+15c^2z^4+6cz^5+z^6.$

16. $a^7+7a^6p+21a^5p^2+35a^4p^3+35a^3p^4+21a^2p^5+7ap^6+p^7.$

17. $m^{11}+11m^{10}n+55m^9n^2+165m^8n^3+330m^7n^4+462m^6n^5$
$+462m^5n^6+330m^4n^7+165m^3n^8+55m^2n^9+11mn^{10}+n^{11}.$

18. $p^{13}+13p^{12}v+78p^{11}v^2+286p^{10}v^3+715p^9v^4+1287p^8v^5+1716p^7v^6$
$+1716p^6v^7+1287p^5v^8+715p^4v^9+286p^3v^{10}+78p^2v^{11}+13pv^{12}+v^{13}.$

19. $a^4-4a^3b+6a^2b^2-4ab^3+b^4.$

20. $a^5-5a^4b+10a^3b^2-10a^2b^3+5ab^4-b^5.$

21. $a^7-7a^6x+21a^5x^2-35a^4x^3+35a^3x^4-21a^2x^5+7ax^6-x^7.$

22. $h^{13}+13h^{12}z+78h^{11}z^2+286h^{10}z^3+715h^9z^4+1287h^8z^5+1716h^7z^6$
$+1716h^6z^7+1287h^5z^8+715h^4z^9+286h^3z^{10}+78h^2z^{11}+13hz^{12}+z^{13}.$

23. $h^{11}-11h^{10}z+55h^9z^2-165h^8z^3+330h^7z^4-462h^6z^5+462h^5z^6$
$-330h^4z^7+165h^3z^8-55h^2z^9+11hz^{10}-z^{11}.$

24. $a^8-8a^7c+28a^6c^2-56a^5c^3+70a^4c^4-56a^3c^5+28a^2c^6-8ac^7+c^8.$

25. $b^9-9b^8p+36b^7p^2-84b^6p^3+126b^5p^4-126b^4p^5+84b^3p^6-36b^2p^7$
$+9bp^8-p^9.$

26. $z^{13}-13z^{12}d+78z^{11}d^2-286z^{10}d^3+715z^9d^4-1287z^8d^5+1716z^7d^6$
$-1716z^6d^7+1287z^5d^8-715z^4d^9+286z^3d^{10}+78z^2d^{11}+13zd^{12}-d^{13}.$

27. $1+12x+66x^2+220x^3+495x^4+792x^5+924x^6+792x^7+495x^8$
$+220x^9+66x^{10}+12x^{11}+x^{12}.$

28. $1-9x+36x^2-84x^3+126x^4-126x^5+84x^6-36x^7+9x^8-x^9.$

29. $1-8a+28a^2-56a^3+70a^4-56a^5+28a^6-7a^2+a^8.$

30. $1+15a+105a^2+455a^3+1365a^4+3003a^5+5005a^6+6435a^7$
$+6435a^8+5005a^9+3003a^{10}+1365a^{11}+455a^{12}+105a^{13}+15a^{14}+a^{15}.$

31. $1-14z+91z^2-364z^3+1001z^4-2002z^5+3003z^6-3432z^7$
$+3003z^8-2002z^9+1001z^{10}-364z^{11}+91z^{12}-14z^{13}+z^{14}.$

32. $x^9+9x^8+36x^7+84x^6+126x^5+126x^6+84x^3+36x^2+9x+1.$

33. $x^{11}-11x^{10}+55x^9-165x^8+330x^7-462x^6+462x^5-330x^4$
$+165x^3-55x^2+11x-1.$

34. $y^8-8y^7+28y^6-56y^5+70y^4-56y^3+28y^2-8y+1.$

35. $z^7+7z^6+21z^5+35z^4+35z^3+21z^2+7z+1.$

36. $z^8-16z^7+112z^6-448z^5+1120z^4-1792z^3+1792z^2-1024z+256.$

37. $8+12x+6x^2+x^3.$ 38. $16-32x+24x^2-8x^3+x^4.$

39. $6561+17496x+20412x^2+13608x^3+5670x^2+1512x^5+252x^6$
$+24x^7+x^8.$

40. $x^9-27x^8+324x^7-2268x^6+10206x^5-30618x^4+61236x^3$
$-78732x^2+59049x-19683.$

41. $z^6+30z^5+375z^4+2500z^3+9375z^2+18750z+15625.$

42. $a^8 - 40a^7 + 700a^6 - 7000a^5 + 43750a^4 - 175000a^3 + 437500a^2$
$- 625000a + 390625$.
43. $16384 - 28672b + 21504b^2 - 8960b^3 + 2240b^4 - 336b^5 + 28b^6 - b^7$.
44. $h^{10} - 20h^9 + 180h^8 - 960h^7 + 3360h^6 - 8064h^5 + 13440h^4$
$- 15360h^3 + 11520h^2 - 5120h + 1024$.
45. $a^6 - 60a^5 + 1500a^4 - 20000a^3 + 150000a^2 - 60000a + 1000000$.
46. $b^5 + 20b^4 + 160b^3 + 640b^2 + 1280b + 1024$.
47. $32 + 80x + 80x^2 + 40x^3 + 10x^4 + x^5$.
48. $243 - 405z + 270z^2 - 90z^3 + 15z^4 - z^5$.

Ex. 20.

1. 66.
2. 581.
3. -128.
4. -214.
5. -21.7.
6. -1.23.
7. -11.
8. $43\frac{3}{17}$.
9. $-.1123$.
10. $-.022\frac{7}{81}$.
11. .3183+.
12. .0101321+.

Ex. 21.

5. $3m$.
6. $-4a^3$.
7. $\frac{5a}{c}$.
8. $-\frac{1}{x^2}$.
9. $6a$.
10. $7ac$.
11. $\frac{x}{5y}$.
12. $-9a^4$.
13. $\frac{-3bm}{4ax}$.
14. bc^2.
15 m^4x^2.
16. $-17ay$.
17. $\frac{9m}{y}$.
18. $-\frac{6y^2}{x}$.
19. $\frac{4mx^4}{5a^3}$.
20. $6x^2z$.
21. $\frac{3bc^3d^2}{a^2}$.
22. $\frac{3am^3n}{pq^3}$.
23. $8ab^2z^2$.
24. $-14x^4y^4z^6$.
25. $8bx^3$.
26. a^3bx.
27. $17a^3$.
28. $7a^{n-6}$.

Ex. 22.

3. $-2am + 3bn - 4cp$.
4. $-7a + 6b - 5c$.
5. $3x^4 - 2x^2 + 1$.
6. $x - 2x^3 + 3x^5 - 4x^7$.
7. $-5m^2 - 4my + 2y^2$.
8. $2a^2 - 3ab + 6b^2$.
9. $-4xy^2 + 5x^2y + 8x^3$.
10. $x^3y^3 - 2x^2 + 3xy - 1$.
11. $3 - 2ab - a^2b^2$.
12. $-z - 2xz^2 + 5x^2yz^2 - 6x^3y^2$.
13. $4ac^3 - 2a^2c^2 + 3a^3bc$.

Ex. 23.

6. $x - 4$.
7. $x - 8$.
8. $x^2 + x + 3$.
9. $3x^2 - 2x + 4$.
10. $3x^2 + 2x + 1$.
11. $x^2 - 3x + 7$.
12. $x^5 + x^4 + x^3 + x^2 + x + 1$.
13. $a^2 + ab - b^2$.
14. $x^3 + 3x^2y + 9xy^2 + 27y^3$.
15. $x^4 + x^3y + x^2y^2 + xy^3 + y^4$.
16. $a^4 - 2a^3b + 4a^2b^2 - 8ab^3 + 16b^4$.
17. $2a^3 - 6a^2b + 18ab^2 - 27b^3$.

18. x^2-2x+2.
19. x^2-3x-1.
20. x^2-5x+6.
21. x^2-4x+8.
22. x^2-x-19.
23. $1-3x+2x^2-x^3$.
24. $x^4+2x^3+3x^2+2x+1$.
25. $a^2+2ab+3b^2$.
26. $2x^3-3x^2+2x$.
27. $a^4+3a^3+9a^2+27a+81$.
28. $6x^2-7x+8$.
29. x^2-3x-y.
30. $x^2-3xy-2y^2$.
31. x^2+xy+y^2.
32. x^2+xy-y^2.
33. $x-y-z$.
34. $4-6x+8x^2-10x^3$.
35. $x+y$.
36. $x^2-3xy-y^2$.
37. $4x^2+2xy+y^2$.
38. $-8a^3+2a^2b-ab^2$.
39. $1+x-2x^2$.
40. $-1-3ab-13a^2b^2$.
41. $x^4+2x^3y+2x^2y^2+xy^3$.
42. $x^3-3x^2y+3xy^2-y^3$.
43. $a^4+a^2b^2+3b^4$.
44. $27x^2y-18xy^2-9y^3$.
45. $a^2+2ab+4b^2$.
46. $x^4+3x^3y+8x^2y^2-8y^4$.
47. $4a^2+4ab+3b^2$.
48. $a^2+b^2+c^2-ab-ac-bc$.
49. $a+c+2b$.
50. $a^2+2ab+b^2-ac-bc+c^2$.

Ex. 24.

1. y^2+y+1.
2. $b^2+5b+25$.
3. $a^2+6a+36$.
4. $x^2+7x+49$.
5. $x^4+x^3y+x^2y^2+xy^3+y^4$.
6. $a^4+a^3+a^2+a+1$.
7. $1+2x+4x^2$.
8. $x^4+2x^3b+4x^2b^2+8xb^3+16b^4$.

Ex. 25.

1. x^2-xy+y^2.
2. $x^4-x^3y+x^2y^2-xy^3+y^4$.
3. $1-2a+4a^2$.
4. $9a^2-3ab+b^2$.
5. $4a^2x^2-2ax+1$.
6. $x^2-3xy+9y^2$.
7. $a^4-2a^3b+4a^2b^2-8ab^3+16b^4$.
8. $64x^2y^2-8xyz+z^2$.

Ex. 26.

1. $x^3+x^2y+xy^2+y^3$.
2. $x^3-x^2y+xy^2-y^3$.
3. $a^5+a^4x+a^3x^2+a^2x^3+ax^4+x^5$.
4. $a^5-a^4x+a^3x^2-a^2x^3+ax^4-x^5$.
5. $x^3+3x^2y+9xy^2+27y^3$.
6. $x^3-3x^2y+9xy^2-27y^3$.
7. $8x^3+4x^2+2x+1$.
8. $8x^3-4x^2+2x-1$.
9. $27a^3x^3+9a^2x^2+3ax+1$.
10. $27a^3x^3-9a^2x^2+3ax-1$.

Ex. 27.

1. $x^4-x^2y^2+y^4$.
2. a^4-a^2+1.
3. $a^8-a^6y^2+a^4y^4-a^2y^6+y^8$.
4. $b^8-b^6+b^4-b^2+1$.
5. $a^8-a^4b^4+b^8$.
6. x^8-x^4+1.
7. $16x^4-4x^2y^2+y^4$.
8. $16-4a^2+a^4$.

Ex. 28.

1. $x=4$.
2. $x=2$.
3. $x=2$.
4. $x=5$.
5. $x=1$.
6. $x=1$.
7. $x=6$.
8. $x=9$.
9. $x=7$.
10. $x=11$.
11. $x=9$.
12. $x=2$.
13. $x=-7$.
14. $x=3$.
15. $x=7$.
16. $x=0$.
17. $x=8$.
18. $x=6$.
19. $x=4$.
20. $x=10$.
21. $x=3$.
22. $x=15$.
23. $x=2$.
24. $x=3$.
25. $x=4$.

Ex. 29.

6. 70.
7. 23.
8. 21, 7.
9. 36, 26, 18, 12.
10. 8, 12.
11. Father, 50 yrs.; son, 30 yrs.
12. 30, 26, 22, 18, 14, 10.
13. $68.
14. Cloth, $3; silk, $6.
15. 52.
16. A, $130; B, $150; C, $130; D, $90.
17. 152 men; 76 women; 38 children.
18. 13, 21.
19. A, 84 yrs.; B, 26 yrs.
20. $21.
21. 50, 40.
22. 42, 18.
23. 24, 60.
24. A, 88 years; B, 44 years.
25. 18 years.
26. 40 years.
27. A, 57 years; B, 19 years.
28. 10 dollar pieces; 40 twenty-five-cent pieces.
29. 28 pounds of better kind; 2 pounds of poorer kind.
30. 28 days.
31. 40 children.
32. $31\frac{1}{2}$ gallons.
33. 2 half-dollars; 6 quarter-dollars; 24 dimes; 48 half-dimes.

Ex. 30.

1. $5a(a-3)$.
2. $6a(a^2+3a-2)$.
3. $7(7x^2-3x+2)$.
4. $4xy(x^2-3xy+2y^2)$.
5. $y(y^3-ay^2+by+c)$.
6. $3a^3b^2(2a^2b-7a+9b^2)$.
7. $27x^2y^6(2+4x^2y^2-9x^4y^3)$.
8. $45x^4y^7(x^3y^3-2x-8y)$.
9. $70ay^4(a^2-2ay+3y^2)$.
10. $32a^3b^6(1+3a^3b^2-4a^5b^3)$.

Ex. 31.

1. $(x-a)(x-b)$.
2. $(a-y)(b+y)$.
3. $(b-x)(c+x)$.
4. $(m+a)(x+n)$.
5. $(cx+y)(dx-y)$.
6. $(ab+pq)(x-y)$.
7. $(cx+ay)(dx-by)$.
8. $(ac-bd)(by-dx)$.
9. $(a-b)(x-y)$.
10. $(cz+y)(dz-y)$.

Ex. 32.

1. $(x+8)(x+3)$.
2. $(x+6)(x+5)$.
3. $(y+12)(y+5)$.
4. $(z+12)(z+1)$.
5. $(x+11)(x+10)$.
6. $(y+20)(y+15)$.
7. $(b+17)(b+6)$.
8. $(x+2)(x+1)$.
9. $(x+6)(x+1)$.
10. $(a+8b)(a+b)$.

Ex. 33.

1. $(x-5)(x-2)$.
2. $(x-19)(x-10)$.
3. $(a-12)(a-11)$.
4. $(b-20)(b-10)$.
5. $(z-23)(z-20)$.
6. $(x-6)(x-1)$.
7. $(x^2-3a^2)(x^2-a^2)$.
8. $(x-6)(x-2)$.
9. $(z-56)(z-1)$.
10. $(y^3-4)(y^3-3)$.

Ex. 34.

1. $(x+7)(x-1)$.
2. $(x+12)(x-7)$.
3. $(y+12)(y-5)$.
4. $(y+15)(y-3)$.
5. $(z+12)(z-1)$.
6. $(z+20)(z-7)$.
7. $(a+25)(a-12)$.
8. $(a+30)(a-5)$.
9. $(b^4+4)(b^4-1)$.
10. $(bc+14)(bc-11)$.

Ex. 35.

1. $(x-7)(x+4)$.
2. $(y-9)(y+2)$.
3. $(x-12)(x+3)$.
4. $(z-15)(z+4)$.
5. $(z-14)(z+1)$.
6. $(a-20)(a+5)$.
7. $(c^5-10)(c^5+1)$.
8. $(x-10)(x+2)$.
9. $(y-10a)(y+5a)$.
10. $(ab-4)(ab+1)$.

Ex. 36.

1. $(x+6)^2$.
2. $(x+14)^2$.
3. $(x+17)^2$.
4. $(z+1)^2$.
5. $(y+100)^2$.
6. $(z^2+7)^2$.
7. $(x+18y)^2$.
8. $(y^2+8z^2)^2$.
9. $(y^3+12)^2$.
10. $(2a+3b^2)^2$.

Ex. 37.

1. $(a-4)^2$.
2. $(a-15)^2$.
3. $(x-19)^2$.
4. $(x-20)^2$.
5. $(y-50)^2$.
6. $(y^2-10)^2$.
7. $(y-25z)^2$.
8. $(x^2-16y^2)^2$.
9. $(z^3-17)^2$.
10. $(2x^2y-5y^2z)^2$.

Ex. 38.

1. $(a+b)(a-b)$.
2. $(a+4)(a-4)$.
3. $(2a+5)(2a-5)$.
4. $(a^2+b^2)(a+b)(a-b)$.
5. $(a^2+1)(a+1)(a-1)$.
6. $(a^4+b^4)(a^2+b^2)(a+b)(a-b)$.

7. $(a^4+1)(a^2+1)(a+1)(a-1)$.
8. $(6x+7y)(6x-7y)$.
9. $(10xy+11ab)(10xy-11ab)$.
10. $(1+7x)(1-7x)$.
11. $(a^2+5b)(a^2-5b)$.
12. $(a-b+c)(a-b-c)$.
13. $(x+a-b)(x-a+b)$.
14. $(a+b+c+d)(a+b-c-d)$.
15. $4xy$.
16. $(1+a-b)(1-a+b)$.
17. $(x+y+z)(x-y-z)$.
18. $(x-y+z)(x-y-z)$.
19. $(a+2b-3c)(a-2b+3c)$.
20. $(a-y+x+z)(a-y-x-z)$.
21. $(z+x-y)(z-x+y)$.
22. $(x-y+z+d)(x-y-z-d)$.
23. $(x-z+y-a)(x-z-y+a)$.
24. $(a+b+c)(a+b-c)$.
25. $(a-b+x-y)(a-b-x+y)$.
26. $(ax+by+1)(ax+by-1)$.
27. $(1+x-y)(1-x+y)$.
28. $(a-b+x)(a-b-x)$.
29. $(a+b+c)(a-b-c)$.
30. $(2x^2+3x-1)(2x^2-3x+1)$.

Ex. 39.

1. $(a-b)(a^2+ab+b^2)$.
2. $(x-2)(x^2+2x+4)$.
3. $(x-7)(x^2+7x+49)$.
4. $(y-5)(y^2+5y+25)$.
5. $(y-6)(y^2+6y+36)$.
6. $(2x-3y)(4x^2+6xy+9y^2)$.
7. $(4y-10z)(16y^2+40yz+100z^2)$.
8. $(9x-8y)(81x^2+72xy+64y^2)$.
9. $(3a-12)(9a^2+36a+144)$.
10. $(10a-11b)(100a^2+110ab+121b^2)$.

Ex. 40.

1. $(x+y)(x^2-xy+y^2)$.
2. $(x+2)(x^2-2x+4)$.
3. $(x+6)(x^2-6x+36)$.
4. $(y+4z)(y^2-4yz+16z^2)$.
5. $(4b+5c)(16b^2-20bc+25c^2)$.
6. $(6a+8c)(36a^2-48ac+64c^2)$.
7. $(9x+12y)(81x^2-108xy+144y^2)$.
8. $(x+y)(x^4-x^3y+x^2y^2-xy^3+y^4)$.
9. $(x+y)(x^6-x^5y+x^4y^2-x^3y^3+x^2y^4-xy^5+y^6)$.
10. $(2b+3c)(16b^4-24b^3c+36b^2c^2-54bc^3+81c^4)$.

Ex. 41.

1. $(a^2+b^2)(a^4-a^2b^2+b^4)$.
2. $(a^2+2b)(a^8-2a^6b+4a^4b^2-8a^2b^3+16b^4)$.
3. $(x^4+y^4)(x^8-x^4y^4+y^8)$.
4. $(b^2+4c^2)(b^4-4b^2c^2+16c^4)$.
5. $(x^2+1)(x^4-x^2+1)$.
6. $(a^4+1)(a^8-a^4+1)$.
7. $(4a^2+x)(16a^4-4a^2x+x^2)$.
8. $(9+c^2)(81-9c^2+c^4)$.

Ex. 42.

1. $(a+b)^3$.
2. $(a+1)^3$.
3. $(a-1)^3$.
4. $(x+y)^4$.
5. $(x-1)^4$.
6. $(a-c)^4$.
7. $(x+y+z)^2$.
8. $(x-y-z)^2$.
9. $(a+b-c)^2$.

Ex. 43.

1. $5(x+1)(x-4)$.
2. $2x^3(x-2)(x-6)$.
3. $3(ab+1)(ab-4)$.
4. $(a+x)(a+x+4)$.
5. $(a-b+c)(a-b-c)$.
6. $(x-y+c-d)(x-y-c+d)$.
7. $(a+b)(a-b-1)$.
8. $(x-y)(x+y-z)$.
9. $(a-b)(b-c)$.
10. $(3x-y)(x-z)$.
11. $(a+x)(a-x-b)$.
12. $(a-x)(a-x+1)$.
13. $(x-y)(3x+3y-2)$.
14. $x(x^2+1)(x+1)$.
15. $(ax-1)(a^3x^3-ax-1)$.
16. $(x+y)(x-y)(x^2+xy+y^2)(x^2-xy+y^2)$.
17. $(x^2+y^2)(x^4-x^2y^2+y^4)$.
18. $y(x^4+y^4)(x^8-x^4y^4+y^8)$.
19. $c(a^2+c^2)(a+c)(a-c)$.
20. $(x+7)(x-3)$.
21. $3(a-2b)(a-5b)$.
22. $(2a-b)^2$.
23. $4(2x-5y)^2$.
24. $x^3y^2(6a+5bx)(6a-5bx)$.
25. $(3xy^2-5z)^2$.
26. $x(4x^2+1)(2x+1)(2x-1)$.
27. $(x-y-z)^2$.
28. $(1+x^2)(1-x)$.
29. $(x+13)(x+7)$.
30. $(x+3)(x-8)$.
31. $(x+y-z)(x-y+z)(x+y+z)(x-y-z)$.
32. $(x^2+1)(3x-1)$.
33. $(x-m+n)(x-m-n)$.
34. $(a+b+c)(a+b-c)(c+a-b)(c-a+b)$.
35. $a^5(a^2+1)$.
36. $(1-7a^3x)^2$.
37. $(y+9)(y-13)$.
38. $(x+15)(x-9)$.
39. $(2a-3b+2c)(2a-3b-c)$.
40. $(a-b)^3$.
41. $(x+y)^3$.
42. $(m+n)(m-n)(p-q)$.
43. $2x(x+7)(x-5)$.
44. $2x(2a-x)(4a^2+2ax+x^2)$.
45. $4b(2x-y)(4x^2+2xy+y^2)$.
46. $x(1-3x)(1+3x+9x^2)$.
47. $(x^2+y^2)(x+y)(x-y)(x^4-x^2y^2+y^4)$. $(x^2-xy+y^2)(x^2+xy+y^2)$.
48. $(7m+11n)(7m-11n)$.
49. $(4+9y^2)(2+3y)(2-3y)$.
50. $(x^2+1)(x-1)$.
51. $(x+y+1)(x-y+1)$.
52. $(x-8)(x-45)$.
53. $5x(5x^2+7y^2)^2$.
54. $(a+2b-3c-d)(a-2b+3c-d)$.

Ex. 44.

1. $18abcd$.
2. $17pq$.
3. $4x^2y^2z^2$.
4. $30x^2y^3$.
5. $a-b$.
6. $a-x$.
7. $a+x$.
8. $3x+1$.
9. $7x-4$.
10. $2a^2xy(3x-y)$.
11. $2abc$.
12. $x-3$.

13. $2(a+b)$.
14. $6x^2(x-y)$.
15. $3x(x+4)$.
16. $c(a-b)$.
17. $5y$.
18. $x(x+1)$.
19. $3(x-1)$.
20. $2(a-b)$.
21. $x+y$.
22. $x-y$.
23. $x-1$.

Ex. 45.

1. $5x-1$.
2. $x+1$.
3. $a(3a-1)$.
4. $(3x+2)(3x-2)$.
5. $3x(3x^2-x+1)$.
6. $2(x-2)$.
7. $2x-5$.
8. $a(a-2x)$.
9. $2x^2-3x+3$.
10. $2x-7$.
11. $5x(6x-1)$.
12. $y(3x-4y)$.
13. $3x-5$.
14. $4x+1$.
15. $3(x+3)$.
16. x^2+xy+y^2.
17. $a-2$.
18. $2(2y+5)$.
19. x^3-5x+1.
20. $7x+1$.

Ex. 46.

1. $x+1$.
2. $y-1$.
3. x^2-2x+5.
4. $x-2$.
5. y^2-2y+5.
6. $x-1$.
7. x^2+x+1.
8. $x+1$.
9. $2(x+y)$.
10. x^2-xy+y^2.

Ex. 47.

1. $12a^3x^2$.
2. $72ax^2y^2$.
3. $x^2(a+x)$.
4. $x(x+1)(x-1)$.
5. $a(a+b)(a-b)$.
6. $(2x+1)(2x-1)$.
7. a^3+b^3.
8. $(x^2+1)(x+1)(x-1)$.
9. $x(x^3+1)(x^3-1)$.
10. $x(x+1)(x^3-1)$.
11. $(2a-1)(8a^3+1)$.
12. $(a-b)(a+b)^2$.
13. $4(1+x)(1-x)$.
14. x^3-1.
15. $(x+y)^2(x-y)^2$.
16. $12(x-y)^2(x^3+y^3)$.
17. $120xy(x+y)(x-y)$.
18. $(x+2)(x+3)(x+4)$.
19. $(a-3)(a+4)(a-5)$.
20. $(x+5)(x+6)(x+7)$.
21. $(x+2)(x-2)(x-11)$.
22. $240(x+1)(x-1)(x+2)(x-2)$.
23. $12x^2y^2(x+y)^2(x-y)^2$.
24. $(a-b)(b-c)(c-a)$.
25. $(a-b)(a-c)(b-c)$.
26. $12xy(x-y)^3$.
27. $(a+b+c+d)(a+b-c-d)(a+c-b-d)(a+d-b-c)$.
28. $60(x-3)(x-2)^2$.

Ex. 48.

1. $(2x+1)(3x-2)(7x-1)$.
2. $(x+1)(x-1)(x+3)(2x+1)(3x-2)$.
3. $(x-12)(x-3)(x^2-2)(x^2+3x+9)$.

4. $(5x-1)(2x+3)(x+4)$.
5. $2(3x-2y)(3x+5y)(4x+3y)$.
6. $2x(x^2+1)(x-1)(x^2-x-1)$.
7. $12(2x+1)(2x-1)(3x+2)(x-3)(x-4)$.
8. $(x-1)(x-2)(x-3)(x-4)$.
9. $(x+y)(x-y)(x+2y)(x-2y)$. 10. $1+p^2+p^4$.
11. $(1-a)^3$. 12. $(a+b+c)(a-b+c)(a+b-c)(b+c-a)$.
13. $(b-1)(b^3-b^2-8)(4b^2-8b+1)$.

Ex. 49.

1. $\frac{x-1}{4x}$.
2. $\frac{x-5}{x-3}$.
3. $\frac{x+1}{x-7}$.
4. x^2-x+1.
5. $\frac{x^3+y^3}{x^3-y^3}$.
6. $\frac{a^2-a+1}{a^2+a+1}$.
7. $\frac{a-5}{a-3}$.
8. $\frac{x-2}{x+4}$.
9. $\frac{x-3}{x+1}$.
10. $\frac{x^2+xy-y^2}{x^2-xy-y^2}$.
11. $\frac{a^2+5a+5}{a^2+a-2}$.
12. $\frac{3x-1}{x^2-1}$.
13. $\frac{x^2-2x+2}{x^2-2}$.
14. $\frac{2x-3a}{4x^2+6ax+9a^2}$.
15. $\frac{5a+2b}{3a+2b}$.
16. $\frac{a-b-c}{a+b-c}$

Ex. 50.

1. $x-1$.
2. $3x-10+\frac{41}{x+4}$.
3. $3x-6+\frac{29}{x+4}$.
4. $a-2x+\frac{3x^2}{a+x}$.
5. $2x+6+\frac{23}{x-3}$.
6. $2a-3x+\frac{7x^2}{5a-x}$.
7. $12x+3+\frac{19}{4x-1}$.
8. $2x+3+\frac{10}{x-4}$.
9. $a+b+\frac{2b^2}{a-b}$
10. $x-1+\frac{5x+4}{5x^2+4x-1}$.

Ex. 51.

1. $\frac{2y}{x+y}$.
2. $\frac{2x}{x+y}$.
3. $\frac{x^2-1}{x}$.
4. $\frac{2(a^2-ax+x^2)}{a-x}$.
5. $\frac{2(11a^2-20ab+8b^2)}{5a-6b}$.
6. $\frac{2ab}{a+b}$.
7. $\frac{2(19a-23a^2-1)}{5-6a}$.
8. $\frac{ax+3}{2a}$.
9. $\frac{2a}{a-b}$.

10. $\frac{-2b}{a+b}$.
11. $\frac{x^2-2xy-y^2}{x+y}$.
12. $\frac{29a}{4}$.
13. $\frac{a^2}{a+1}$.
14. $\frac{x^2}{x-3}$.
15. $\frac{2a^2-ab-b^2}{a+b}$.
16. $\frac{3x^2+2x+1}{x+4}$.
17. $\frac{x^3+1}{x-1}$.
18. $\frac{x(x^3-2x^2-3)}{x-2}$.
19. $\frac{a^3+2x^3}{a+2x}$.
20. $\frac{x^2+xy+y^2}{x+a}$.

Ex. 52.

1. $\frac{9x-21}{18}, \frac{4x-9}{18}$.
2. $\frac{4x-8y}{10x^2}, \frac{3x^2-8xy}{10x^2}$.
3. $\frac{48a^2-60ac}{60a^2c}, \frac{15a-10c}{60a^2c}$.
4. $\frac{5+5x}{1-x^2}, \frac{6}{1-x^2}$.
5. $\frac{a-c}{(a-b)(a-c)(b-c)}, \frac{b-c}{(a-b)(a-c)(b-c)}$.
6. $\frac{8x^2(a-b)}{6(a^2-b^2)}, \frac{xy}{6(a^2-b^2)}$.
7. $\frac{30(4x+1)}{15(x-2)}, \frac{5(2x-1)}{15(x-2)}, \frac{3(3x+2)}{15(x-2)}$.
8. $\frac{an-bmn}{mnx}, \frac{mnx}{mnx}, \frac{cm-bmn}{mnx}$.

Ex. 53.

1. $\frac{16x^2+55x+4xy-55y}{50x}$.
2. $\frac{27x^2-2x^2y-16xy-28y^2}{12x^2}$.
3. $\frac{180a^2+54ab-20ab^2+331b^2}{90b^2}$.
4. $\frac{80x^3+64x^2+84x+45}{60x^2}$.
5. $\frac{32x+9y}{42}$.
6. $\frac{11y^2-7x^2-4xy-8x^2y^2}{x^3y^3}$.
7. $\frac{b^2c^2-2ab^2c+2a^2bc-a^2c^2}{a^2b^2c^2}$.
8. $\frac{6a^3-3a^4-2}{8a^2}$.
9. $\frac{a^2b+b^2c+ac^2}{abc}$.
10. $-\frac{1}{6y^2z}$.

Ex. 54.

1. $\frac{2x-1}{x^2-x-30}$.
2. $\frac{4}{x^2-10x+21}$.
3. $\frac{2}{1-x^2}$.
4. $-\frac{1}{1+x}$.
5. $\frac{2x-y}{(x-y)^2}$.
6. $\frac{1}{a^2-x^2}$.

7. $\dfrac{a^3-a^2b-ab^2-b^3}{ab(a^2-b^2)}$.

8. $\dfrac{7x-17}{4x(x^2-3x+2)}$.

9. $\dfrac{2x^3}{1+x^2+x^4}$.

10. $\dfrac{2a-3b}{x^2-y^2}$.

Ex. 55.

1. $\dfrac{2}{1-a}$.

2. $\dfrac{4x}{1-x^4}$.

3. $\dfrac{2x+x^3}{1+x^2}$.

4. $\dfrac{x+y}{y}$.

5. $\dfrac{3x^2-2ax-6a^2}{(x-a)^3}$.

6. $\dfrac{6}{(x^2-1)(x+2)}$.

7. 0.

8. 2.

9. $\dfrac{y}{x+y}$.

10. 0.

11. $\dfrac{ax}{(a-x)(a+2x)}$.

12. $\dfrac{9a-10b+c}{(a-b)(a-c)(b-c)}$.

13. $\dfrac{-2(x^2-xy+y^2)}{xy(x-y)}$.

14. $\dfrac{2x^3-5x^2y+10xy^2+5y^3}{(x+y)^2(x-y)^2}$.

15. $\dfrac{ac-ab+b^2-c^2}{(a+b+c)(a+b-c)(a+c-b)}$.

Ex. 56.

1. $\dfrac{y}{x-y}$.

2. $\dfrac{1}{2+x}$.

3. $\dfrac{3x^2}{x^2-1}$.

4. $\dfrac{6+y}{3(1+y)(1-y)}$.

5. 0.

6. $\dfrac{1}{(x+a)(x+b)}$.

7. $\dfrac{a^6+2a^5b-2ab^5+b^6}{(a^3+b^3)(a^3-b^3)}$.

8. $\dfrac{b^2-ax}{b^2-x^2}$.

9. $\dfrac{1}{2+x}$.

10. 0.

11. 0.

12. 0.

13. 0.

14. $\dfrac{9a-10b+c}{(a-b)(a-c)(b-c)}$.

15. $\dfrac{1}{z(z-x)(y-z)}$.

Ex. 57.

1. $\dfrac{ac}{bd}$.

2. $9ax$.

3. $\frac{3}{4}$.

4. $\dfrac{4xy}{5b}$.

5. $\dfrac{by}{9ax}$.

6. $-ax$.

7. $-\dfrac{15x}{4z}$.

8. $\dfrac{3mnxy}{4pq^2}$.

9. $\dfrac{5km^2}{4pq}$.

10. $\dfrac{a-b}{a^2}$.

11. $\dfrac{a^2+b^2}{(a-b)^2}$.

12. $\dfrac{(x-1)(x-6)}{x^2}$.

13. $\frac{x-6}{x-3}$.
14. $\frac{(a+x)(a^2+ax+x^2)}{(a-x)(a^2-ax+x^2)}$.
15. $\frac{2ax^2(x-y)}{c}$.
16. $\frac{ab}{a^2+4b^2}$.
17. $\frac{(x-2)(x-5)}{x^2}$.
18. $\frac{x}{x^2+y^2}$.
19. $-\frac{m+n}{c^2-cd+d^2}$.
20. 1.
21. b.
22. $\frac{y}{x-y}$.
23. $\frac{3a}{2b}$.
24. $\frac{c-a+b}{c-a-b}$.
25. $\frac{x-a+b}{x+a-b}$.
26. 1.
27. $\frac{x-y-z}{x+y+z}$.

Ex. 58.

1. $\frac{x-4}{x-5}$.
2. $\frac{1}{x+1}$.
3. $\frac{(x+a)(x-a)}{ax+bx+cx-bc}$.
4. $\frac{xy^2}{(x+y)(x-y)^2}$.
5. $\frac{x^2+1}{2x}$.
6. $\frac{1}{x+1}$.
7. $\frac{1+x}{1+x^2}$.
8. $x+1$.
9. $\frac{1+x^2}{1+x}$.
10. x.
11. $\frac{a(a^2+2ab+2b^2)}{(a+b)^2}$.
12. $m-1$.

Ex. 59.

1. $\frac{x-8}{x+8}$.
2. $1\frac{3}{5}$.
3. 46.
4. $\frac{1}{2(x+1)^2}$.
5. x.
6. 0.
7. 1.
8. $\frac{xy}{x^2+y^2}$.
9. 2.
10. $x^2+3x+3-\frac{3}{x}+\frac{1}{x^2}$.
11. 0.
12. a.
13. $\frac{1}{abc}$.
14. m.
15. $\frac{1}{(x+1)^2}$.

Ex. 60.

1. $x=16$.
2. $x=5$.
3. $x=\frac{1}{2}$.
4. $x=1$.
5. $x=8$.
6. $x=-\frac{1}{5}$.
7. $x=9$.
8. $x=1\frac{1}{2}$.
9. $x=7$.
10. $x=-5$.
11. $x=3$.
12. $x=10$.
13. $x=6$.
14. $x=12$
15. $x=5$.
16. $x=2$.
17. $x=7$.
18. $x=9$.
19. $x=9$.

Ex. 61.

1. $x = 8$. 3. $x = 4$. 5. $x = 1\frac{1}{2}$. 7. $x = 2$. 9. $x = -2$.
2. $x = 9$. 4. $x = 20$. 6. $x = 4$. 8. $x = 3$. 10. $x = 11$.

Ex. 62.

1. $x = c$. 6. $x = 2$. 11. $x = \frac{18a + 2b}{3 + 4a}$.
2. $x = \frac{3c - 2a}{5b - c}$. 7. $x = 0$. 12. $x = b - 1$.
3. $x = \frac{c - d}{a^2 - b^2 + b - c}$. 8. $x = \frac{b}{a - 1}$. 13. $x = 0, \frac{b}{c}$.
4. $x = 1$. 9. $x = \frac{b}{c}$. 14. $x = bm$.
5. $x = -\frac{(a + b)^2}{a - b}$. 10. $x = \frac{3a + 1}{2a + b}$.

Ex. 63.

1. $x = 7$. 4. $x = 3$. 7. $x = a + \frac{b^2}{a + b}$. 9. $x = 0$.
2. $x = -\frac{11}{12}$. 5. $x = 0$. 10. $x = 1$.
3. $x = 1\frac{1}{5}$. 6. $x = -2$. 8. $x = c$. 11. $x = 1$.

Ex. 64.

1. 24. 3. 80. 5. 28, 32. 7. 76, 24. 9. 3456, 2304.
2. 168. 4. 8. 6. $2\frac{6}{7}$, $47\frac{1}{7}$. 8. 960. 10. 420.
11. 85 gallons wine, 35 gallons water.
12. 18 pounds saltpetre, 3 pounds sulphur, 3 pounds charcoal.
13. 28, 18. 14. House, $600; garden, $250.
15. $7200. 17. 47, 23. 19. 19, 20, 21, 22.
16. $\frac{a + b}{2}$, $\frac{a - b}{2}$. 18. 7, 32. 20. 42 years.
21. 20 years. 24. 2 days.
22. Father, 36 years; son, 12 years. 25. $1\frac{1}{29}$ days.
23. A, 28 years; B, 14 years. 26. $26\frac{2}{3}$ days.
27. B and C, $21\frac{9}{11}$ days; A, B, and C, $10\frac{10}{23}$ days.
28. 10 days. 32. 48 minutes.
29. 40 minutes. 33. 42 hours.
30. A, $37\frac{1}{2}$ minutes; B, 25 minutes. 34. 7 miles.
31. $4\frac{4}{5}$ hours. 35. 160 miles.
36. First, 5 miles per hour; second, 3 miles per hour.
37. 175 miles.

38. I. $16\frac{4}{11}$ minutes past three; II. $32\frac{8}{11}$ minutes past six; III. $49\frac{1}{11}$ minutes past nine.
39. I. $32\frac{8}{11}$ minutes past three; II. $5\frac{5}{11}$ and $38\frac{2}{11}$ minutes past four; III. $21\frac{9}{11}$ and $54\frac{6}{11}$ minutes past 7.
40. I. $38\frac{2}{11}$ minutes past one; II. $54\frac{6}{11}$ minutes past four; III. $10\frac{10}{11}$ minutes past eight.
41. 700 leaps.
42. Greyhound, 450 leaps; hare, 600 leaps.
43. 240 leaps.
44. Length, 13 feet; breadth, 8 feet.
45. $56\frac{1}{4}$ square feet.
46. Length, 27 in.; breadth, 14 in.
47. Length, 15 feet; breadth, 11 feet.
48. 111 pounds tin; 69 pounds lead.
49. 76 pounds gold; 30 pounds silver.
50. $6\frac{2}{3}$ ounces.
51. 330 feet.
52. 24 shots.
53. 60.
54. 20 seconds.
55. $55,500.

Ex. 65.

1. $x = 2.$ $y = 1.$
2. $x = 2.$ $y = -1.$
3. $x = 2.$ $y = 8.$
4. $x = 12.$ $y = -3.$
5. $x = 6.$ $y = 7.$
6. $x = 7.$ $y = 9.$
7. $x = 29.$ $y = 23.$
8. $x = 8.$ $y = 4.$
9. $x = 7.$ $y = 14.$
10. $x = 3.$ $y = 2.$
11. $x = 3.$ $y = 20.$
12. $x = -3.$ $y = -2.$
13. $x = -2.$ $y = 1.$
14. $x = 7.$ $y = -2.$

Ex. 66.

1. $x = 10.$ $y = 7.$
2. $x = 17.$ $y = 19.$
3. $x = 2.$ $y = 13.$
4. $x = 4.$ $y = 1.$
5. $x = 5.$ $y = 5.$
6. $x = 21.$ $y = 12.$
7. $x = 20.$ $y = 10.$
8. $x = 2.$ $y = -3.$
9. $x = 8.$ $y = 4.$
10. $x = 2.$ $y = 2.$
11. $x = 2.$ $y = 3.$
12. $x = 9.$ $y = -3.$
13. $x = -5.$ $y = 14.$
14. $x = \frac{1}{2}.$ $y = -\frac{1}{3}.$

Ex. 67.

1. $x = 8.$ $y = 3.$
2. $x = 6.$ $y = 3.$
3. $x = 6.$ $y = 8.$
4. $x = 2.$ $y = 13.$
5. $x = 8.$ $y = 1.$
6. $x = 3.$ $y = 4.$
7. $x = 12.$ $y = 3.$
8. $x = 2.$ $y = 6.$

9. $x=3.$ $y=5.$

10. $x=4.$ $y=3.$

11. $x=-2.$ $y=19.$

12. $x=7.$ $y=-5.$

13. $x=2.$ $y=-3.$

14. $x=11.$ $y=10.$

Ex. 68.

1. $x=1.$ $y=7.$
2. $x=2.$ $y=3.$
3. $x=1.$ $y=8.$
4. $x=9.$ $y=8.$
5. $x=-3.$ $y=-2.$
6. $x=7.$ $y=10.$
7. $x=11.$ $y=6.$
8. $x=14.$ $y=46.$
9. $x=12.$ $y=6.$
10. $x=9.$ $y=8.$
11. $x=5.$ $y=7.$
12. $x=7.$ $y=2.$
13. $x=4.$ $y=-1.$
14. $x=6.$ $y=5.$

Ex. 69.

1. $x=\frac{a+b}{2}.$ $y=\frac{a-b}{2}.$

2. $x=\frac{cq-br}{aq-bp}.$ $y=\frac{ar-cp}{aq-bp}.$

3. $x=\frac{bn-aq}{np-mq}.$ $y=\frac{ap-bm}{np-mq}.$

4. $x=\frac{ce-bd}{a(c-b)}.$ $y=\frac{e-d}{b-c}$

5. $x=\frac{nr'+n'r}{mn'+m'n}.$ $y=\frac{mr'-m'r}{mn'+m'n}.$

6. $x=\frac{c(f-bc)}{af-bd}.$ $y=\frac{c(ac-d)}{af-bd}.$

7. $x=\frac{ab^2c^2}{b^2c-a^2}.$ $y=\frac{a^2bc}{a^2-b^2c}.$

8. $x=\frac{1}{ab}.$ $y=\frac{1}{cd}.$

9. $x=\frac{2b^2-6a^2+d}{3a}.$ $y=\frac{3a^2-b^2+d}{3b}.$

10. $x=\frac{a}{a-b}.$ $y=\frac{b}{a+b}.$

Ex. 70.

1. $x=\frac{1}{2}.$ $y=\frac{1}{4}.$

2. $x=\frac{1}{b-2a}.$ $y=\frac{2}{3a-b}.$

3. $x=6.$ $y=9.$

4. $x=\frac{1}{2}.$ $y=1.$

5. $x=-1.$ $y=-\frac{1}{2}.$

6. $x=\frac{ab(a+b)}{c(a^2+ab+b^2)}.$ $y=-\frac{a+b}{c}.$

7. $x=\frac{1}{a}.$ $y=\frac{1}{b}.$

8. $x = \frac{1}{n}$. $y = \frac{1}{m}$.

9. $x = \frac{a^2 + b^2}{am + bn}$. $y = \frac{a^2 + b^2}{bm - an}$.

Ex. 71.

1. $x = 2$. $y = 2$. $z = 2$.
2. $x = 5$. $y = 6$. $z = 8$.
3. $x = 1$. $y = 2$. $z = 3$.
4. $x = 5$. $y = 6$. $z = 7$.
5. $x = 1$. $y = 4$. $z = 6$.
6. $x = 2$. $y = 9$. $z = 10$.
7. $x = 20$. $y = 10$. $z = 5$.
8. $x = \frac{2}{3}$. $y = -7$. $z = 36\frac{1}{3}$.
9. $x = 2$. $y = 3$. $z = 1$.
10. $x = 4$. $y = 0$. $z = 5$.
11. $x = 5$. $y = -5$. $z = 5$.
12. $x = 45$. $y = -21$. $z = 1$.
13. $x = 1$. $y = 2$. $z = -3$.
14. $x = 1$. $y = -1$. $z = 2$.
15. $x = -1\frac{1}{2}$. $y = 2\frac{1}{2}$. $z = 6\frac{1}{2}$.
16. $x = -4$. $y = -3\frac{2}{3}$. $z = -4\frac{1}{2}$.
17. $x = \frac{1}{3}$. $y = \frac{1}{4}$. $z = \frac{1}{6}$.
18. $x = 9$. $y = 11$. $z = 13$.
19. $x = 6$. $y = -12$. $z = 18$.
20. $x = 1$. $y = \frac{1}{2}$. $z = \frac{1}{3}$.
21. $x = \frac{2}{a + b}$; $y = \frac{2}{a + c}$; $z = \frac{2}{b + c}$.
22. $x = 3\frac{1}{3}$; $y = 1\frac{3}{7}$; $z = \frac{10}{11}$.
23. $x = 1\frac{1}{6}$; $y = -3\frac{1}{2}$; $z = 2\frac{1}{10}$.

Ex. 72.

1. 41, 7.
2. 6, 4.
3. Father, 35 years; son, 14 years.
4. A, \$116; B, \$166.
5. Wheat, \$$1\frac{1}{4}$ per bushel; barley, \$$\frac{3}{4}$ per bushel.
6. A, \$31; B, \$27.
7. Horse, \$100; cow, \$50.
8. $\frac{5}{6}$.
9. $\frac{9}{11}$.
10. $\frac{35}{41}$.
11. $\frac{8}{15}$.
12. $\frac{4}{5}$, $\frac{5}{6}$.
13. 28.
14. 24.
15. 84.
16. 75.
17. 36.
18. 69.
19. 717.
20. 759.
21. 126.
22. 432.
23. $7\frac{1}{2}$ hours, $4\frac{1}{2}$ hours.
24. 3 miles per hour.
25. Distance, 20 miles; rate per hour, 8 miles.
26. Time rowing down, 4 hours; time rowing up, 6 hours; rate of stream per hour, $\frac{5}{6}$ miles.
27. 20 pounds @ 42 cents; 10 pounds @ 54 cents.
28. 40 pounds @ 90 cents; 90 pounds @ 28 cents.
29. Rye, 20 bushels; wheat, 52 bushels.
30. A, \$$3\frac{2}{3}$; B, \$3; C, \$$2\frac{1}{3}$.
31. A, $1\frac{3}{4}$ hours; B, $3\frac{1}{2}$ hours; C, 7 hours.
32. 54 boxes; 36 bales.
33. 1st, 36 days; 2d, 45 days.
34. 21 crowns; 63 guineas.
35. A, 20 days; B, 30 days; C, 60 days.

36. A, $\frac{2abc}{ac+bc-ab}$ min.; B, $\frac{2abc}{ab+bc-ac}$ min.; C, $\frac{2abc}{ab+ac-bc}$ min.
37. \$5000 @ 5%; \$5000 @ 4%.
38. \$20,000 @ 5%. 39. \$25,000 @ 6%.
40. Sum, $\frac{an-bm}{n-m}$ dollars; rate, $\frac{100(b-a)}{an-bm}$%.
41. Sum, $\frac{ad-bc}{a-b}$ dollars; rate, $\frac{1200(c-d)}{ad-bc}$%.
42. 1st, \$3000 @ 4%; 2d, \$4000 @ 5%; 3d, \$4500 @ 6%.
43. 2660 square yards. 44. Length, 14 feet; breadth, 10 feet.
45. Length, 90 yards; breadth, 60 yards.

Ex. 73.

1. a^6. 2. x^{15}. 3. x^4y^6. 4. $\frac{a^{12}b^8}{16}$. 5. $\frac{243x^{10}y^5}{32a^{15}b^{10}}$.
6. $16a^8b^4c^{12}$.
7. $-125a^3x^9y^6$.
8. $49m^6n^2x^4y^8$.
9. $-\frac{32x^{15}y^5}{243a^5b^5c^5}$.
10. $-243a^{10}b^{10}c^5$.
11. $729x^6y^{18}$.
12. $-3125a^{10}b^5x^{15}$.
13. $\frac{81a^4b^8}{256c^{12}}$.
14. $-\frac{x^{14}y^{21}z^{28}}{128}$.
15. $x^3+6x^2+12x+8$.
16. $x^4-8x^3+24x^2-32x+16$.
17. $x^5+15x^4+90x^3+270x^2+405x+243$.
18. $1+10x+40x^2+80x^3+80x^4+32x^5$.
19. $8m^3-12m^2+6m-1$.
20. $81x^4+108x^3+54x^2+12x+1$.
21. $16x^4-32ax^3+24a^2x^2-8a^3x+a^4$.
22. $243x^5+810ax^4+1080a^2x^3+720a^3x^2+240a^4x+32a^5$.
23. $16x^4-32x^3y+24x^2y^2-8xy^3+y^4$.
24. $x^{12}y^6-12x^{11}y^7+60x^{10}y^8-160x^9y^9+240x^8y^{10}-192x^7y^{11}+64x^6y^{12}$.
25. $a^7b^7-21a^6b^6+189a^5b^5-945a^4b^4+2835a^3b^3-5103a^2b^2+5103ab-2187$.
26. $1-2a-a^2+2a^3+a^4$.
27. $1-6x+15x^2-20x^3+15x^4-6x^5+x^6$.
28. $1-3x+6x^2-7x^3+6x^4-3x^5+x^6$.

Ex. 74.

1. $\pm a^2$, $\pm x^2$, $\pm 2a^3b$, 4, ax^2y^3, $\pm 2a^3bc^2$, $-2a^3$.
2. $-12c^3d^4xy^3$, $15b^7z^5$, $\pm 42c^4z$.
3. $\pm 231b^2c^4y^6z^8$, $-\frac{6bc^5}{7z^8}$, $\pm\frac{2x^3}{3z^5}$.
4. $2ab^2c$.
5. $36x^3y^3z^2$.
6. 92.
7. 170.
8. 16.
9. 1.

Ex. 75.

1. $a^2 + 2a - 1$.
2. $x^2 - xy + y^2$.
3. $2a^3 - 3a^2x - ax^2$.
4. $3x^3 - 4xy^2 - 2y^3$.
5. $2a^4 + 4a^2c^2 - 4c^4$.
6. $2x^2 - 5x + 3$.
7. $4x^2 - 2ab + 2b^2$.
8. $x^3 - 2x^2 + 3x - 4$.
9. $x^3 - 2x^2y + 2xy^2 - y^3$.
10. $2 - 3a - a^2 + 2a^3$.
11. $5x^3 - 3x^2y - 4xy^2 + y^3$.
12. $x^2 - \frac{1}{2}xy - y^2$.
13. $x^2 - 2xy + y^2 - \dfrac{y^3}{x}$.

Ex. 76.

1. 347; 69.4; 43.21; 37.89; 8.008.
2. 129.63; 2.1319; .4937; .75416.
3. .9486.....; 2.4919.....; .6557.....; .0923.....; 4.1231.....; 11.3578.....; 18.6348.....
4. 119.5633.....; 1.5811.....; 44.7213.....; .5477.....; .1732.....; 10.5356.....
5. .0333.....; .0632.....; .0707.....; 1.4142.....; 2.2360.....; 1.8027.....; 2.9325.....
6. $\frac{1}{2}$; $\frac{4}{7}$; $\frac{5}{6}$; $\frac{13}{15}$; $\frac{17}{18}$; $\frac{4}{5}$.
7. .7071.....; .8164.....; .8660.....; .1767.....; .2338.....; .2190.....; .9258.....; .2886.....

Ex. 77.

1. $x + 2y$.
2. $a - 3$.
3. $x + 4$.
4. $x^2 - ax - a^2$.
5. $x^2 + x + 1$.
6. $1 - 3x + 4x^2$.
7. $a^2 - 2a - 1$.
8. $4x^2 + 4x - 1$.
9. $1 - x + x^2 - x^3$.
10. $a^2 + 3ab - 9b^2$.
11. $c^2 - 4bc + 4b^2$.
12. $2a^2 + 4ab - 3b^2$.

Ex. 78.

1. 65.
2. 48.
3. 64.
4. 9.6.
5. 478.
6. 114.
7. 11.7.
8. 108.
9. 1.41.
10. 192.
11. 2.34.
12. 3.84.
13. 32.1.
14. 46.8.
15. 9.36.
16. 20.53.
17. 1.111.
18. 27.55.
19. 4.5045.
20. 1.7479.
21. 1.3572.....; .5848.....; .2154.....; 1.5874.....; .7368.....

Ex. 79.

1. $3a - 5b$.
2. $1 - x + x^2$.
3. $2 - x$.
4. $3x - 1$.
5. $1 - y$.

Ex. 80.

1. ± 7.
2. ± 3.
3. $\pm \sqrt{3}$.
4. $\pm \frac{1}{2}$.
5. $\pm \frac{1}{2}$.
6. $\pm \sqrt{11}$.
7. ± 1.
8. ± 5.

Ex. 89.

1. 4, 2, $3 \pm \sqrt{21}$.
   2, 4, $3 \mp \sqrt{21}$.
2. 3, 2, $-3 \pm \sqrt{3}$.
   2, 3, $-3 \mp \sqrt{3}$.
3. 4, -2, 0.
   2, -4, 0.
4. 4, 2, $\frac{1}{6}(-13 \pm \sqrt{377})$.
   2, 4, $\frac{1}{6}(-13 \mp \sqrt{377})$.
5. ± 3, ± 2.
   ± 2, ± 3.
6. 3, 2.
   2, 3.

Ex. 90.

1. 4, 3.
   -3, -4.
2. 8, -3.
   $1\frac{1}{2}$, -4.
3. ± 7, ± 1.
   ± 1, ± 7.
4. 2, $-2\frac{9}{19}$.
   -1, $-2\frac{15}{19}$.
5. 8, 1.
   -1, -8.
6. $\pm \frac{4}{5}$.
   $\pm \frac{4}{7}$.
7. $2\frac{1}{2}$, $-1\frac{1}{2}$.
   $1\frac{1}{2}$, $-2\frac{1}{2}$.
8. 4.
   -4.
9. $\frac{1}{4}(\pm 3 \pm \sqrt{29})$.
   $\frac{1}{4}(\pm 3 \mp \sqrt{29})$.
10. $\dfrac{a}{2} \pm b$.
    $\dfrac{a}{2} \mp b$.
11. ± 4, $\pm 10\frac{5}{8}$.
    ± 9, $\pm 2\frac{3}{8}$.
12. 4, 2.
    2, 4.
13. 10, $-10\frac{7}{9}$.
    15, $-16\frac{1}{6}$.
14. 3, -2.
    2, -3.
15. 3, -2.
    2, -3.
16. 12, 10; 10, 12.
17. $\frac{1}{2}(a \pm \sqrt{2b^2 - a^2})$; $\frac{1}{2}(a \mp \sqrt{2b^2 - a^2})$.
18. $\pm 1\frac{1}{2}$, $\pm \frac{1}{2}\sqrt{3}$. $\frac{1}{2}\{a - b \pm \sqrt{(a + 3b)(a - b)}\}$.
    $\pm 1\frac{1}{2}$, $\pm \frac{1}{2}\sqrt{3}$. $\frac{1}{2}\{a - b \mp \sqrt{(a + 3b)(a - b)}\}$.
19. ± 3, ± 2.
    ± 2, ± 3.
20. a, 0.
    0, $-a$.
21. ± 4, ± 3.
    ± 3, ± 4.
22. $6\frac{1}{2}$.
    $1\frac{1}{2}$.

Ex. 91.

1. 7, 5.
2. 34, 43.
3. $\frac{1}{2}(3 \pm \sqrt{5})$, $\frac{1}{2}(1 \pm \sqrt{5})$.
4. 16, 10.
5. $\frac{2}{3}$, $\frac{3}{4}$.
6. 15 miles; 3, $2\frac{1}{2}$, and 4 miles.
7. 73, 37.
8. 16, 12.
9. 88 yards; 55 yards.
10. 8 feet; 10 feet.

Ex. 92.

1. $x^{\frac{3}{2}}$; $x^{\frac{2}{3}}$; $x^{\frac{5}{2}}$; $a^{\frac{4}{3}}$; $a^{\frac{6}{7}}$; $a^{\frac{7}{5}}$; $a^{\frac{1}{2}}b^{\frac{1}{3}}$.
2. $x^{\frac{1}{3}}y^{\frac{2}{3}}z$; $x^{\frac{3}{5}}y^{\frac{2}{5}}z^{\frac{4}{5}}$; $a^{\frac{5}{7}}b^{\frac{6}{7}}c$; $5ab^{\frac{1}{2}}c^{\frac{3}{2}}x^2$.
3. $\sqrt[3]{a^2}$; $\sqrt[6]{a^3b}$; $4\sqrt[6]{xy^{-5}}$; $3\sqrt[3]{xy^{-2}}$.
4. $\dfrac{1}{a^2}$; $\dfrac{3}{xy^3}$; $\dfrac{6y}{x^3}$; $\dfrac{x^4}{y^5}$; $\dfrac{6xy^3}{ab^2}$.
5. $3xyz^{-2}$; $x^{-3}y^{-4}z$; $ab^{-1}c^{-1}$; $a^{-3}b^2c^2$; $x^{-\frac{1}{3}}y^{\frac{2}{3}}$; $x^{-2}y^{-\frac{1}{3}}$.
6. $a^{\frac{5}{6}}$; $b^{\frac{1}{2}}$; $c^{\frac{3}{4}}$; $d^{\frac{2}{3}}$.
7. $m^{\frac{1}{3}}$; $n^{\frac{2}{3}}$; $a^{\frac{1}{2}}$; $\dfrac{1}{a^{\frac{1}{2}}}$.
8. a; 1; $y^{\frac{1}{2}}$; $x^{\frac{1}{3}}$.
9. $a^{\frac{1}{2}}b^{\frac{3}{2}}c^{\frac{4}{3}}$; $ac^{\frac{1}{4}}d$.
10. $\dfrac{y^{\frac{1}{6}}}{x^{\frac{1}{3}}z^{\frac{1}{3}}}$; $\dfrac{x^{\frac{1}{2}}}{y^{\frac{1}{4}}}$.
11. $\dfrac{1}{a^{\frac{5}{2}}}$; $\dfrac{y^{\frac{1}{3}}}{b^{\frac{1}{3}}x^{\frac{1}{6}}}$.
12. $a^{\frac{1}{6}}$; $c^{\frac{1}{5}}$; $n^{-\frac{1}{6}}$; $a^{\frac{1}{6}}$.
13. a^{-1}; $c^{-\frac{1}{6}}$; m^{-2}; n^{-1}; x.
14. $p^{\frac{1}{2}}$; $q^{-\frac{1}{6}}$; $x^{\frac{1}{6}}y^{-1}$; $a^{-\frac{4}{3}}$.

15. $\frac{a}{8}$; $\frac{b^2}{9}$; $\frac{1}{32c^{\frac{10}{3}}}$; $\frac{4}{c^4}$.

16. $\frac{27a^3b^{\frac{9}{4}}}{8}$; $\frac{64}{27a^6b^{\frac{9}{2}}}$; $\frac{a^2}{3^{\frac{1}{4}}}$; $\frac{125}{64}$.

Ex. 93.

1. $x^{4p} + x^{2p}y^{2p} + y^{4p}$.
2. $x^{mn} - x^n y^n + x^{mn-n}y^{mn-n} - y^{mn}$.
3. $x - 3x^{\frac{2}{3}} + 3x^{\frac{1}{3}} - 1$.
4. $16a + 8a^{\frac{6}{7}}b^{\frac{1}{7}} + 10a^{\frac{5}{7}}b^{\frac{2}{7}} + 18a^{\frac{4}{7}}b^{\frac{3}{7}} - 8a^{\frac{3}{7}}b^{\frac{4}{7}} - 4a^{\frac{2}{7}}b^{\frac{5}{7}} - 5a^{\frac{1}{7}}b^{\frac{6}{7}} - 9b$.
5. $1 + a^2b^{-2} + a^4b^{-4}$.
6. $a^4b^{-4} - 4a^{-2}b^2 - a^{-4}b^4$.
7. $4x^{-5} - x^{-4} + 3x^{-3} + 2x^{-2} + x^{-1} + 1$.
8. $x^{3n} + x^{2n}y^n + x^n y^{2n} + y^{3n}$.
9. $x^{\frac{2}{3}} - x^{\frac{1}{3}}y^{\frac{1}{3}} - x^{\frac{1}{3}}z^{\frac{1}{3}} + y^{\frac{2}{3}} - y^{\frac{1}{3}}z^{\frac{1}{3}} + z^{\frac{2}{3}}$.
10. $x^{\frac{1}{3}} + y^{\frac{1}{3}}$.
11. $xy^{-1} + x^{-1}y$.
12. $a^{-2} + a^{-1}b^{-1} + b^{-2}$.
13. $16a^2b^{-2}$; $a - 2a^{\frac{1}{2}}b^{\frac{1}{2}} + b$; $a^2 + 2 + a^{-2}$; $4ab^{\frac{2}{3}} - 4b + a^{-1}b^{\frac{4}{3}}$.
14. 4; 10; 4; $\frac{1}{4}$; $\frac{3}{32}$.
15. $a^{\frac{3}{2}} - 3ab^{\frac{1}{3}} + 3a^{\frac{1}{2}}b^{\frac{2}{3}} - b$.
16. $3x^{-2} - 3x^{-1}y^{\frac{1}{2}} + y$.
17. $2x + 1 - 3x^{-1}$.
18. 24.
19. x^{5b}
20. x^6.
21. $2a^{\frac{1}{2}}b^{\frac{1}{2}} + 11b$.
22. a^{m-1}.

Ex. 94.

1. $\sqrt{45}$; $\sqrt{189}$; $\sqrt{800}$; $\sqrt{a^4b^3c}$; $\sqrt[3]{x^5y^2}$.
2. $\sqrt[4]{81x^3y^9}$; $\sqrt[5]{32x^6y}$; $\sqrt[4]{a^{15}b^2}$; $\sqrt[3]{27abc^7}$; $\sqrt{25a^3b^3c}$.
3. $\sqrt{1\frac{1}{4}}$; $\sqrt{244\frac{4}{9}}$; $\sqrt{xy}$.
4. $xy^2\sqrt{z}$; $2a\sqrt{2ab}$; $3ay\sqrt[3]{2ax^2}$; $2\sqrt{6}$; $5a^2d\sqrt{5d}$.
5. $10\sqrt[3]{a}$; $2xy^2\sqrt[3]{20xy}$; $3m^3n^3\sqrt[3]{4n}$; $7a^5b^5\sqrt[3]{4b}$.
6. $(a-b)\sqrt[3]{a}$; $5(a-b)\sqrt{2}$.
7. $4ac\sqrt[4]{5ab^2c^2}$; $42\sqrt{11x}$; $27y\sqrt[3]{3x^2z}$; $55\sqrt{6}$.
8. $\frac{1}{3}\sqrt{5}$; $\frac{3}{4}\sqrt{3}$; $\frac{5}{4}\sqrt{2}$; $\frac{3}{4}\sqrt{58}$; $\sqrt[3]{4}$.
9. $\frac{1}{z}\sqrt[3]{2xy^2z^2}$; $\frac{1}{5}\sqrt[3]{20}$; $\frac{1}{2b}\sqrt[4]{8ab}$; $\frac{a}{2cy^2}\sqrt{3bcxy}$.
10. $2\frac{2}{5}\sqrt{5}$; $\frac{2}{189}\sqrt{21}$; $\frac{x^{\frac{1}{3}}z^{\frac{1}{2}}}{y^{\frac{1}{2}}}$; $\frac{a^{\frac{2}{3}}b^3}{c}$.
11. $abx^{\frac{2}{3}}$; $2a^2b^{\frac{14}{3}}x$; $15a^{\frac{17}{4}}b^3y^{\frac{7}{4}}$.
12. 7.071065; 42.42639; 0.707107; 0.1414213.

Ex. 95.

1. $3\sqrt{7}$.
2. $6\sqrt{7}$; $5\sqrt{10}$; $9\sqrt{3}$.
3. $5\sqrt[3]{3}$; $4\sqrt[3]{4}$; $3\sqrt[3]{5}$.
4. $24\sqrt{3}$; $\sqrt{6}$.
5. $5\frac{25}{49}\sqrt{7}$; 2.
6. $\frac{2}{9}\sqrt{3}$; $1\frac{1}{3}\sqrt{15}$.
7. $\frac{14}{135}\sqrt{30}$.
8. $4\frac{2}{3}$.
9. $\frac{20}{3}\sqrt[3]{4}$.

Ex. 89.

[illegible] $-13 \pm \sqrt{377})$.   5. $\pm 3, \pm 2$.

$\pm 2, \pm 3$.

[illegible] $-13 \mp \sqrt{377})$.   6. 3, 2.

2, 3.

Ex. 90.

9. [illegible] $\sqrt{29})$.   12. 4, 2.

[illegible] $\sqrt{29})$.   2, 4.

10. $\frac{a}{2} = b$.   13. 10, $-10\frac{7}{8}$.

15, $-16\frac{1}{8}$.

$\frac{a}{2} = b$.   14. 3, -2.

2, -3.

11. $\pm 4, \pm 10\frac{2}{3}$.   15. 3, -2.

$\pm 9, \pm 2\frac{2}{3}$.   2, -3.

17. $\frac{1}{2}(a \pm \sqrt{2b^2 - a^2})$; $\frac{1}{2}(a \mp \sqrt{2b^2 - a^2})$.

18. [illegible] $\frac{1}{2}\{a - b \pm \sqrt{(a+3b)(a-b)}\}$.

[illegible] $\frac{1}{2}\{a - b \mp \sqrt{(a+3b)(a-b)}\}$.

20. a, 0.   21. $\pm 4, \pm 3$.   22. $6\frac{1}{4}$.

0, $-a$. $\pm 3, \pm 4$. $1\frac{1}{4}$.

Ex. 91.

6. 15 miles; 3, $2\frac{1}{2}$, and 4 miles.

7. 73, 37.

8. 16, 12.

9. [illegible] yards; [illegible] yards.

10. [illegible] feet; 10 feet.

Ex. 92.

[illegible]

13. [illegible]

Ex. 33

1. [illegible]

3. [illegible]

4. [illegible]

5. [illegible]

7. [illegible]

8. [illegible]

9. [illegible]

11. [illegible]

13. [illegible]

14. [illegible]

15. [illegible]

16. [illegible]

17. [illegible]

Ex. 34

1. [illegible]

2. [illegible]

3. [illegible]

4. [illegible]

5. [illegible]

6. [illegible]

7. [illegible]

8. [illegible]

9. [illegible]

10. [illegible]

11. [illegible]

Ex. 96.

1. $3\sqrt{2}$; $\frac{5}{4}\sqrt[4]{4}$; $2\sqrt[3]{3}$.
2. $\sqrt[3]{\frac{14}{15}}$; $\sqrt{\frac{3}{5}}$.
3. $3\sqrt[3]{7}$; $4\sqrt{2}$; $2\sqrt[3]{22}$.
4. $3\sqrt{19}$; $5\sqrt[3]{2}$; $3\sqrt[3]{3}$.
5. $2ax\sqrt[6]{72ab^5}$; $a\sqrt[20]{a^3x^9y^{19}}$.
6. $\frac{3}{a^2}\sqrt[6]{2a^5b}$; $\frac{1}{b}\sqrt[12]{8a^{11}}$.
7. $\frac{1}{5}\sqrt[6]{225{,}000a^4b^4}$; 4.
8. $\frac{a}{xy}\sqrt[6]{b^5x^5y^2}$.
9. $6-30\sqrt{3}+24\sqrt{10}$.
10. $\frac{1}{80}$; $(2ab)^x$.
11. $a^8b^7\sqrt[7]{b^2}$; $a^{-\frac{3}{8}}b^{\frac{7}{30}}c^{\frac{13}{24}}d^{-\frac{7}{4}}$.

Ex. 97.

1. $29\sqrt{3}$; $30\sqrt{10}+164\sqrt{2}$.
2. $13\sqrt[3]{2}$; $33\sqrt[3]{2}$.
3. $48\sqrt{2}$; 0.
4. $\frac{8}{3}\sqrt{3}$; $\frac{22}{15}\sqrt{15}$.
5. $\left(\frac{a^2}{b^2}-\frac{ac}{bd}-\frac{ad}{bm}\right)\sqrt{bc}$; $\frac{3}{5}\sqrt{10}$.
6. $(a+10b)\sqrt{ab}$.
7. $abc\sqrt[5]{ab^2c^3}$.
8. 0.
9. $8a^2\sqrt{3b}$; $9\sqrt[3]{3}$.
10. $\frac{1}{3}\sqrt{3a}$; $\sqrt[4]{27}$.
11. $\sqrt[3]{3}$; $\sqrt[4]{8}$; $\frac{1}{2}\sqrt[3]{2}$; $\sqrt[3]{2}$; $\sqrt[4]{3}$.
12. $\sqrt[6]{2}$; $\sqrt[5]{6}$; $\sqrt{2}$; $\sqrt{3}$; $\sqrt[3]{5}$; $\sqrt{7}$.
13. $\sqrt[3]{2x^2}$; $\sqrt[3]{3ab^2}$; $a\sqrt{2a}$; $2a^2$.
14. $2\sqrt[3]{2}$; $3\sqrt[3]{3}$; $4\sqrt[4]{2}$; $2\sqrt[3]{2}$.
15. a^{-4}; $x^{-\frac{2}{3}}$; $p^{\frac{1}{2}}$; $a^{\frac{15}{16}}$.
16. $x^{2m}+3x^my^n+y^{2n}$.
17. $1+2x^{-\frac{1}{3}}-3x^{-\frac{2}{3}}+4x^{-1}$.

Ex. 98.

1. $\frac{3}{2}(\sqrt{7}-\sqrt{5})$; $\frac{1}{2}(2\sqrt{5}+\sqrt{6})$; $5\sqrt{2}-6$; $30+12\sqrt{6}$.
2. $\frac{a(\sqrt{b}+\sqrt{c})}{b-c}$; $\frac{(a+b)(a+\sqrt{b})}{a^2-b}$; $\frac{2(x-y)\sqrt{xy}+3xy}{y(x-4y)}$.
3. 1.154.....; 1.216.....; 3.576.....; 19.726.....

Ex. 99.

1. 19; 11.
2. $6\sqrt{-3}$; $-ax\sqrt{by}$.
3. $b-a$; $-a^4b^4\sqrt{b}$.
4. $-2\sqrt{5}$; $-2(3+13\sqrt{-15})$.
5. $\frac{x}{y}$; $-\sqrt{-1}$; $-\sqrt{-a}$.
6. 2; $-\sqrt{-3}$; $\frac{1}{2}$.

Ex. 100.

1. 9.
2. 4, 0.
3. $2\pm\frac{1}{3}\sqrt{6}$.
4. 6, $2\frac{3}{4}$.
5. 16, 1.
6. 5.
7. 5, $1\frac{24}{121}$.
8. $\frac{25}{36}$.
9. ± 24.
10. ± 5, $\pm 3\sqrt{2}$.
11. $\frac{1}{2}$, $-1\frac{1}{2}$.
12. -28, 0.

13. 12, 3.
14. 1, $\frac{1}{21}$.
15. 1.
16. $\pm\sqrt{65}$, $\pm\sqrt{5}$.
17. $1\frac{1}{4}$.
18. $\pm 2a$.
19. $2a$.
20. $-2a$.
21. $2\sqrt[3]{18}$, 4.
22. $2\frac{1}{2}$, $\frac{2}{5}$.
23. $4a^2$, a^2.
24. $\pm 3\frac{3}{8}$, $\pm\frac{27}{169}\sqrt{-13}$.

Ex. 101.

1. 7, ± 4, -1.
2. $\frac{2}{3}$, $-1\frac{1}{2}$, $\frac{1}{12}(-13\pm\sqrt{313})$.
3. $2\frac{1}{2}$, -1, $\frac{1}{4}(3\pm\sqrt{-15})$.
4. $\frac{a+2b}{2a}$, $\frac{2b-9a}{2a}$.
5. 1, $-\frac{1}{2}$, $\frac{1}{4}(1\pm\frac{1}{3}\sqrt{41})$.
6. 4, 1, $3\frac{2}{3}$, $1\frac{1}{3}$.
7. 1, $\frac{1}{2}(-3\pm\sqrt{5})$.
8. ± 4, $\pm\sqrt{23}$.
9. $\frac{1}{2}(-1\sqrt{5})$, $\frac{1}{2}(-1+\frac{1}{3}\sqrt{58})$.
10. $6\frac{29}{100}$.
11. 4, 1.
12. $1\frac{41}{48}$.
13. $\frac{1}{2}(1\pm\sqrt{-3})$, $\pm\sqrt{-1}$.
14. 5, $-2\frac{1}{2}$, $\frac{1}{4}(5\pm\sqrt{97})$.
15. $\frac{1}{4}$, $-\frac{2}{9}$, $\frac{1}{8}(1\pm\sqrt{33})$.
16. $\frac{4a^2+b^4}{8b^2}$.
17. $\frac{1}{3}$, $-2\frac{1}{3}$, 0.
18. $\pm 2\sqrt{2}$, $\pm\frac{1}{4}\sqrt{-2}$, $\pm\frac{1}{8}\sqrt{(5\pm\sqrt{41})^3}$.
19. $-8\frac{1}{5}$, 5, $\frac{1}{5}(4\mp 3\sqrt{109})$; $6\frac{4}{5}$, -2, $\frac{2}{5}(-3\pm\sqrt{109})$.
20. $\frac{1}{2}(a+b+1)\mp\frac{1}{4}\sqrt{4a+1}\mp\frac{1}{4}\sqrt{4b+1}$;
$\frac{1}{2}(a-b)\mp\frac{1}{4}\sqrt{4a+1}\pm\frac{1}{4}\sqrt{4b+1}$.
21. ± 2, ± 1, $\frac{1}{2}(\pm\sqrt{6}\pm\sqrt{2})$; ± 1, ± 2, $\frac{1}{2}(\pm\sqrt{6}\mp\sqrt{2})$.
22. 6, 2, $\frac{1}{2}(-9\pm\sqrt{33})$; 2, 6, $\frac{1}{2}(-9\mp\sqrt{33})$.
23. $\frac{1}{4b}(a^2+b^2)\pm\frac{1}{4b}\sqrt{10a^2b^2-3a^4-3b^4}$;
$\frac{1}{4b}(a^2+b^2)\mp\frac{1}{4b}\sqrt{10a^2b^2-3a^4-3b^4}$.

Ex. 102.

1. 24 : 35.
2. 10 : 9.
4. 15 : 16; 23 : 25; 10 : 11; 3 : 4.
5. $a+b : a-b > a^2+b^2 : a^2-b^2$.
6. 4 : 15.
7. 18, 27.
9. 6, 8.
10. 4 : 1.
11. 1 : 5.
12. 26, 40, A to B; 52, 60, B to C.
13. 1240, 680 rods.

Ex. 103.

14. 3, $\frac{4}{7}$.
15. $a+b$, $\frac{1}{2}(a-b)$.
16. ± 9; ± 3.
17. 1; 4.
18. ± 2.
19. A's, \$200; B's, \$150.
20. 70 feet.

Ex. 104.

1. 53; 13; -29; $a+21b$; -1; 0.
2. $4\frac{1}{3}$.
3. $7\frac{1}{2}$.
4. $5\frac{1}{2}$, 10, $14\frac{1}{2}$; $\frac{1}{5}$, $4\frac{2}{5}$, $8\frac{3}{5}$, $12\frac{4}{5}$.
5. 25th.
6. -7.
7. 185; 35; $\frac{an}{2}(3n-1)$; -28; $336\frac{2}{3}$.
8. 1, $2\frac{2}{5}$, $3\frac{4}{5}$, $5\frac{1}{5}$, $6\frac{3}{5}$, 8.
9. 9.
10. -10.

11. 6, 14, 22. 12. 6. 13. 5. 14. 3, 5, 7; or 7, 5, 3.
15. 5, 7, 9. 16. $-\frac{3}{4}, \frac{1}{4}, 1\frac{1}{4}$. 17. 10 days. 18. 234.
19. 2, 6, 10, 14; or 14, 10, 6, 2. 20. $6433\frac{1}{3}$ ft.

Ex. 105.

1. 1458; 96; $\frac{3}{128}$; -128; x^{14}; $20\frac{1}{4}a^5m^4$. 2. $6x^2y^2\sqrt{15z}$.
3. 4. 4. 20, 50; 28, 56, 112. 5. 5th. 6. 3; 192.
7. 765; 547; $15\frac{63}{64}$; 1953.1; $\frac{205m}{256}$. 8. 10%. 9. 3. 10. $1\frac{127}{128}$. 12. $\frac{32}{243}$.
13. 12, 18, 27. 14. 1, $2\frac{1}{4}$. 15. 1, 3, 9. 16. 49, 1.
17. 1, 3, 9, 27; or 27, 9, 3, 1. 18. 239.999999 gals.
19. Ratio = 4 or $\frac{1}{4}$; first term = 3 or 192.
20. 8; $1\frac{1}{2}$; $\frac{1}{5}$; $\frac{5}{7}$; $\frac{3}{10}$; $1\frac{1}{5}$; $\frac{1}{9}$; $\frac{86}{99}$; $\frac{49}{90}$; $\frac{46}{55}$.

Ex. 106.

1. $\frac{p^{25}}{32} + \frac{15p^{20}y^4}{16} + \frac{90p^{15}y^8}{8} + \frac{270p^{10}y^{12}}{4} + \frac{405p^5y^{16}}{2} + 243y^{20}$.

2. $\frac{32a^{15}}{243} + \frac{20a^{12}b^2}{27} + \frac{5a^9b^4}{3} + \frac{15a^6b^6}{8} + \frac{135a^3b^8}{128} + \frac{243b^{10}}{1024}$.

3. $a^{10}b^{15} + 10a^{11}b^{13}x^4 + 40a^{12}b^{11}x^8 + 80a^{13}b^9x^{12} + 80a^{14}b^7x^{16} + 32a^{15}b^5x^{20}$.

4. $\frac{128a^7b^{14}}{2187} - \frac{112a^8b^{12}y}{243} + \frac{14a^9b^{10}y^2}{9} - \frac{35a^{10}b^8y^3}{12} + \frac{105a^{11}b^6y^4}{32}$
$- \frac{567a^{12}b^4y^5}{256} + \frac{1701a^{13}b^2y^6}{2048} - \frac{2187a^{14}y^7}{16384}$.

5. $a^3\sqrt{a} + 7a^3x + 21a^2\sqrt{a} \times x^2 + 35a^2x^3 + 35a\sqrt{a} \times x^4 + 21ax^5$
$+ 7\sqrt{a} \times x^6 + x^7$.

6. $16b^4 - 64b^3\sqrt{2b} \times m + 224b^3m^2 - 224b^2\sqrt{2b} \times m^3 + 280b^2m^4$
$- 112b\sqrt{2b} \times m^5 + 56bm^6 - 8\sqrt{2b} \times m^7 + m^8$.

7. $27c^3\sqrt{3c} + 378c^3a + 756c^2\sqrt{3c} \times a^2 + 2520c^2a^3 + 1680c\sqrt{3c} \times a^4$
$+ 2016ca^5 + 448\sqrt{3c} \times a^6 + 128a^7$.

8. $\frac{a^4}{16} - 3a^3\sqrt{\frac{1}{2}a} \times y + \frac{63a^3y^2}{2} - 378a^2\sqrt{\frac{1}{2}a} \times y^3 + \frac{2835a^2y^4}{2}$
$- 6804a\sqrt{\frac{1}{2}a} \times y^5 + 10{,}206ay^6 - 17{,}496\sqrt{\frac{1}{2}a} \times y^7 + 6561y^8$.

9. $64e^3 + 192ae^2\sqrt{e} + 240a^2e^2 + 160a^3e\sqrt{e} + 60a^4e + 12a^5\sqrt{e} + a^6$.

10. $\frac{64a^6}{729}+\frac{64a^5\sqrt{2x}}{81}+\frac{160a^4x}{27}+\frac{320a^3x\sqrt{2x}}{27}+\frac{80a^2x^2}{3}$
$+16ax^2\sqrt{2x}+8x^3.$

11. $\frac{243a^5}{1024}-\frac{405a^4\sqrt{\frac{1}{2}x}}{256}+\frac{135a^3x}{64}-\frac{45a^2x\sqrt{\frac{1}{2}x}}{16}+\frac{15ax^2}{16}-\frac{x^2\sqrt{\frac{1}{2}x}}{4}.$

12. $64a^6-576a^5\sqrt{y}+2160a^4y-4320a^3y\sqrt{y}+4860a^2y^2$
$-2916ay^2\sqrt{y}+729y^3.$

13. $a^{14}+\frac{7a^{12}\sqrt{z}}{2}+\frac{21a^{10}z}{4}+\frac{35a^8z\sqrt{z}}{8}+\frac{35a^6z^2}{16}+\frac{21a^4z^2\sqrt{z}}{32}$
$+\frac{7a^2z^3}{64}+\frac{z^3\sqrt{z}}{128}.$

14. $b^4-8b^3\sqrt{by}+28b^3y-56b^2y\sqrt{by}+70b^2y^2-56by^2\sqrt{by}$
$+28by^3-8y^3\sqrt{by}+y^4.$

15. $8c^3+24c^2\sqrt{6cx}+180c^2x+120cx\sqrt{6cx}+270cx^2+54x^2\sqrt{6cx}+27x^3.$

16. $8{,}200{,}192a^6$. 18. $326{,}592x^{10}$. 20. $\frac{1547a^{11}x^9}{256}$.

17. $352{,}716a^{11}d^{10}$. 19. $-12{,}033{,}222{,}880y^{81}$. 21. $-165a^4x^2$.

22. $(1+x)^{-1}=1-x+x^2-x^3+x^4-x^5+\cdots\cdots$
$1-\frac{1}{9}+\frac{1}{9^2}-\frac{1}{9^3}+\frac{1}{9^4}-\frac{1}{9^5}+\cdots\cdots=0.9.$

23. $(1+x)^{\frac{1}{2}}=1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}-\frac{5x^4}{128}+\frac{7x^5}{256}-\cdots\cdots=1+\frac{3}{16}-\frac{9}{512}$
$+\frac{27}{8192}-\frac{320}{524108}+\frac{1701}{8388608}-\cdots\cdots=1.1726.$

24. $(1+x)^{-\frac{1}{2}}=1-\frac{x}{2}+\frac{3x^2}{8}-\frac{5x^3}{16}+\frac{35x^4}{128}-\frac{63x^5}{256}+\cdots\cdots=0.998503367\cdots\cdots$

25. $a^{-5}-5a^{-6}x+15a^{-7}x^2-35a^{-8}x^3+\cdots\cdots$

26. $a^{-6}+6a^{-7}x+21a^{-8}x^2+56a^{-9}x^3+\cdots\cdots$

27. $\frac{1}{64b^6}+\frac{3y}{64b^7}+\frac{21y^2}{256b^8}+\frac{7y^3}{64b^9}+\cdots\cdots$

28. $\frac{512}{c^9}-\frac{9212z}{c^{10}}+\frac{92160z^2}{c^{11}}-\frac{675840z^3}{c^{12}}+\cdots\cdots$

29. $\frac{1}{a^9}-\frac{9x}{2a^{10}}+\frac{45x^2}{4a^{11}}-\frac{165x^3}{8a^{12}}+\cdots\cdots$

30. $\frac{1}{a^{10}}-\frac{5x^3}{a^{12}}+\frac{15x^6}{a^{14}}-\frac{35x^9}{a^{14}}+\cdots\cdots$

31. $\frac{1}{a^3}+\frac{6x^2}{a^3\sqrt{a}}+\frac{21x^4}{a^4}+\frac{56x^6}{a^4\sqrt{a}}+\ldots\ldots$

32. $\sqrt{b}+\frac{h}{2\sqrt{b}}-\frac{h^2}{8b\sqrt{b}}+\frac{h^3}{16b^2\sqrt{b}}-\ldots\ldots$

33. $\sqrt[4]{b}-\frac{x}{4\sqrt[4]{b^3}}-\frac{3x^2}{32b\sqrt[4]{b^3}}-\frac{7x^3}{128b^2\sqrt[4]{b^3}}-\ldots\ldots$

34. $x+\frac{a}{2x}-\frac{a^2}{8x^3}+\frac{a^3}{16x^5}-\ldots\ldots$

35. $a-\frac{1}{2a}-\frac{1}{8a^3}-\frac{1}{16a^5}-\ldots\ldots$

36. $1+\frac{a}{9}-\frac{4a^2}{81}+\frac{68a^3}{2187}-\ldots\ldots$

37. $\frac{1}{a\sqrt{a}}-\frac{1}{2a^4\sqrt{a}}+\frac{3}{8a^7\sqrt{a}}-\frac{5}{16a^{10}\sqrt{a}}.$

38. $\frac{1}{x}+\frac{a}{2x^3}+\frac{3a^2}{8x^5}+\frac{5a^3}{16x^7}.$

39. $1+\frac{x^5}{3}+\frac{2x^{10}}{9}+\frac{14x^{15}}{81}.$

40. $1-\frac{2d}{5}+\frac{12d^2}{25}-\frac{88d^3}{125}.$

41. $\frac{1}{8}-\frac{3h}{256}+\frac{3h^2}{2048}-\frac{13h^3}{65536}.$

42. $\frac{1}{27}+\frac{x^2}{81}+\frac{5x^4}{1458}+\frac{35x^6}{39366}.$

43. 7.2801098.....

44. 4.431047.....

45. 2.8716219.....

46. 3.0385047.....

47. 4.073062.....

48. 3.87827796.....

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WENTWORTH'S

SERIES OF MATHEMATICS.

ARITHMETIC.

Wentworth & Reed's First Steps in Number.

A Primary Arithmetic.

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The object of this book is to provide teachers with a record of the work done in number in the primary schools of to-day.

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The object of every teacher is so to present numbers that the mind of the child may grasp firmly the facts concerning them, and hold these facts tenaciously by the law of association. Success lies

n requiring the child to *show* what he is talking about, and in following the "step by step" rule. The book illustrates these two principles. It abounds in examples which have not before appeared in print, and which are calculated to interest the child from their close connection with his varied experiences. It gives suggestions for versatility of drill, and illustrates in detail the teaching of a hundred topics.

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It is hoped that this book will find a welcome among all persons interested in leading children by easy and sure paths to a knowledge of numbers.

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Pupils can be trained to logical habits of mind and stimulated to a high degree of intellectual energy by solving problems adapted

to their capacities. They become *practical* arithmeticians, not by learning special business forms, but by founding their knowledge on reasoning which they fully comprehend, and by being so thoroughly exercised in logical analysis that they are independent of arbitrary rules.

This Arithmetic contains a great number of well-graded and progressive problems made up for youths from ten to fourteen years of age. Definitions and explanations are made as brief and simple as possible. It is not intended that definitions should be committed to memory, but that they should be simply discussed by teacher and pupils. Every teacher, of course, will be at liberty to give better definitions, and to make a better presentation of methods, than those exhibited in the book.

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The last three chapters are a short chapter on the Metric System, a chapter on Mensuration, and a chapter of Miscellaneous Problems. The Metric System is treated here because the great majority of grammar-school pupils have no time for the subject, while those who have can as well learn the system at this stage of their progress as earlier. The chapter on Mensuration is suited to the ability of beginners. The intention is not to give a *system* of Geometry, but to render familiar those notions of Geometry that are indispensable for practical purposes. This chapter has been illustrated and enforced by numerous practical examples.

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Introduction, **$1.00**; *Allowance for old book in use,* **30 cts.**

Is intended for high and normal schools and academies. It assumes that the pupil has some knowledge of the simple processes of Arithmetic, and aims to develop his power over practical questions as well as to increase his facility in computing. The shortest road to a thorough acquaintance with the principles of Arithmetic is by solving problems, not by memorizing rules or solving propositions. Hence stereotyped methods and set rules are avoided. Such problems are selected as are calculated to interest the pupil and lead him to independent thought and discovery.

The problems cover a wide range of subjects, and are particularly adapted to general mental discipline, to preparation for higher studies, mechanical work, business or professional life.

Decimal fractions are introduced at an early stage, and abundant practice in operations with them is given by means of the metric system. The chapter on the metric system may be omitted without affecting the unity of the book; but teachers, even if opposed to the substitution of the metric for the ordinary measures, can use this chapter to great advantage as a drill in the decimal system. Experience has shown that the best preparation for learning common fractions and the common measures is a thorough familiarity with decimals.

Percentage in its various applications is fully explained, and is illustrated by many examples, so that the pupil will understand the principles involved, and work intelligently in after life, whether he is required to compute interest, average accounts, etc., directly, or by means of interest tables.

The nature and use of logarithms are briefly treated, and a four-place table of the logarithms of the natural numbers from 1 to 1000 is given for the purpose of saving time and labor in the solution of many practical questions.

Wentworth & Hill's Practical Arithmetic.

Abridged Edition.

Introduction, **75 cts.**; *Allowance for old book in use,* **30 cts.**

This edition is Wentworth and Hill's Practical Arithmetic reduced to 244 pages by the omission of some of the more difficult problems and topics, so as to make it suitable for Grammar Schools.

The Answers are bound in at the end of the book.

ALGEBRA.

Wentworth's Shorter Course in Algebra.

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THE SHORTER COURSE IN ALGEBRA is intended for schools that have not sufficient time for the author's full course. The book, however, contains a full treatment of the topics usually found in an elementary algebra. Care has been taken to exclude all difficult problems, and all necessary aid is given for the statement and solution of the different classes of problems.

The Answers are bound at the end of the volume.

Wentworth's Elements of Algebra.

Introduction, **$1.12**; *Allowance for old book in use,* **40 cts.**

This book is designed for high schools and academies, and contains an ample amount for admission to any college.

The single aim in writing this volume has been to make an Algebra which the beginner would read with increasing interest, intelligence, and power. The fact has been kept constantly in mind that, to accomplish this object, the several parts must be presented so distinctly that the pupil will be led to feel that he is *mastering* the subject. Originality in a text-book of this kind is not to be expected or desired, and any claim to usefulness must be based upon the method of treatment and upon the number and character of the examples. About four thousand examples have been selected, arranged, and tested in the recitation-room, and any found too difficult have been excluded from the book. The idea has been to furnish a great number of examples for practice, but to exclude complicated problems that consume time and energy to little or no purpose.

In expressing the definitions, particular regard has been paid to brevity and perspicuity. The rules have been deduced from processes immediately preceding, and have been written, not to be committed to memory, but to furnish aids to the student in framing for himself intelligent statements of his methods. Each principle has been fully illustrated, and a sufficient number of problems has been given to fix it firmly in the pupil's mind before he proceeds to another. Many examples have been worked out in order to exhibit the best methods of dealing with different classes of problems and the best arrangement of the work; and such aid has been given in the statement of problems as experience has shown to be necessary for the attainment of the best results. General demonstrations have been avoided whenever a particular illustration would serve the purpose, and the application of the principle to similar cases was obvious. The reason for this course is, that the pupil must become familiar with the separate steps from particular examples, before he is able to follow them in a general demonstration, and to understand their logical connection.

Wentworth's Complete Algebra.

Introduction, **$1.40;** *Allowance for old book in use,* **40 cts.**

This work is the continuation of the Elementary Algebra (described above), and contains about 150 pages more than that. The additions are chapters on Chance, Interest Formulas, Continued Fractions, Theory of Limits, Indeterminate Coefficients, the Exponential Theorem, the Differential Method, the Theory of Numbers, Imaginary Numbers, Loci of Equations, Equations in General, and Higher Numerical Equations.

Wentworth & McLellan's University Algebra.

[*In Preparation.*

GEOMETRY.

Wentworth's Plane Geometry.

Introduction, **75 cts.;** *Allowance for old book in use,* **25 cts.**

Wentworth's Plane and Solid Geometry.

Introduction, **$1.25;** *Allowance for old book in use,* **40 cts.**

This work is based upon the assumption that Geometry is a branch of practical logic, the object of which is to detect and state precisely the successive steps from premise to conclusion.

In each proposition a concise statement of what is given is printed in one kind of type, of what is required in another, and the demonstration in still another. The reason for each step is indicated in small type, between that step and the one following, thus preventing the necessity of interrupting the process of demonstration by referring to a previous proposition. The number of the section, however, on which the reason depends, is

placed at the side of the page; and the pupil should be prepared, when called upon, to give the proof of each reason.

A limited use has been made of symbols, wherein symbols stand for words, and not for operations.

Great pains have been taken to make the page attractive. The propositions have been so arranged that in no case is it necessary to turn the page in reading a demonstration.

A large experience in the class-room convinces the author that, if the teacher will rigidly insist upon the *logical form* adopted in this work, the pupil will avoid the discouraging difficulties which usually beset the beginner in Geometry; that he will rapidly develop his reasoning faculty, acquire facility in simple and accurate expression, and lay a foundation of geometrical knowledge which will be the more solid and enduring from the fact that it will not rest upon an effort of the memory simply.

Wentworth's Plane and Solid Geometry, and Plane Trigonometry.

Introduction, **$1.40**; *Allowance for old book in use,* **40 cts.**

TRIGONOMETRY AND TABLES.

Wentworth's Plane Trigonometry.

Paper. *Introduction,* **30 cts.**

Wentworth's Plane Trigonometry and Tables.

Introduction, **60 cts.**

Wentworth's Plane and Spherical Trigonometry, Surveying, and Tables.

Introduction, **$1.25**; *Allowance for old book in use,* **40 cts.**

Wentworth's Plane and Spherical Trigonometry, Surveying, and Navigation.

Introduction, **$1.12**; *Allowance for old book in use,* **40 cts.**

As this work is intended for beginners, an effort has been made to develop the subject in the most simple and natural way.

In the first chapter the functions of an acute angle are defined as ratios, and the fundamental relations of the functions are established and illustrated by numerous examples. It is afterwards shown how the numerical values of the ratios may be represented by lines, and the simpler line-values are employed in studying the changes of the functions as the angle changes.

In the second chapter the right triangle is solved, and many problems are given in order that the student may at the outset perceive the practical utility of Trigonometry, and acquire skill in the use of logarithms.

In the third chapter the definitions of the functions are extended to all angles, and the necessary propositions are established by simple proofs.

In the fourth chapter the oblique triangle is solved, and a collection of miscellaneous examples is added.

The fifth chapter treats of the right spherical triangle; the sixth, of the oblique spherical triangle; and the seventh contains many interesting problems in Geometry and Astronomy to be solved by applying the principles and formulas of Spherical Trigonometry.

The object of the work on Surveying and Navigation is to present these subjects in a clear and intelligible way, according to the best methods in actual use; and also to present them in so small a compass, that students in general may find the time to acquire a competent knowledge of these very interesting and important studies.

The answers to the problems in Trigonometry, Surveying, and Navigation, are printed at the end of the separate parts.

Wentworth & Hill's Five-Place Tables.

(Seven Tables.)

Introduction, **50 cts.**

Wentworth & Hill's Five-Place Tables.

(Complete Edition.)

Introduction, **$1.00.**

Table I. contains the common logarithms of the natural numbers from 1 to 10,000.

Table II. contains the values of π, its most useful combinations, and the corresponding logarithms.

Table III. contains the logarithms of the trigonometric functions of angles from 0° to 0° 3′ and from 89° 57′ to 90° for every second; from 0° to 2° and from 88° to 90° for every ten seconds; and from 1° to 89° for every minute.

Table IV. gives a method of working with great accuracy when the angle lies between 0° and 2° or 88° and 90°.

Table V. contains the natural sines, cosines, tangents, and cotangents to four decimal places, and at intervals of ten minutes.

Table VI. contains the values of the circumference and area of a circle for different values of the radius, and of the radius and area for different values of the circumference.

Table VII. gives the latitude and departure to three places of decimals for distances from 1 to 10 corresponding to bearings from 0° to 90° at intervals of fifteen minutes.

These seven tables are sufficient for solving problems in Trigonometry and Surveying. The complete edition contains eighteen tables, all that are required for solving problems in Trigonometry, Surveying, and Navigation.

The tables are preceded by an introduction, in which the nature and use of logarithms are explained, and all necessary instruction given for using the tables.

The seven tables occupy 60 pages, and the complete tables occupy 140 pages. They are printed in large type with very open spacing. Compactness, simple arrangement, and figures large enough not to strain the eyes, are secured by excluding proportional parts from the tables. These are considerations of the very highest importance, and it is doubtful whether the printing of proportional parts has any advantage for the purposes of instruction where the main object is to inculcate principles. Experience shows that beginners without the aid of proportional parts learn in a very short time to interpolate with great rapidity and accuracy.

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